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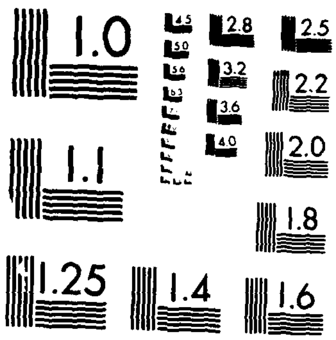

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INTEGRAL MANIFOLDS OF SLOW ADAPTATION

Bradley Dean Riedle

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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A three-step procedure for the analysis of slowly adapting systems is presented. First, conditions are given for the existence of an integral manifold upon which the slow adaptation occurs in either continuous or discrete time. Second, conditions are derived under which the integral manifold is exponentially attractive. Third, the behavior on the manifold is analyzed via the method of averaging. In the process of developing the discrete-time part of these results, the relationship between the method of averaging for deterministic signals and the ordinary differential equation approach to the study of stochastic adaptive systems is clarified.

This three-step procedure for analysis is then used as a design tool. First, a model reference adaptive control scheme which allows a reduced number of adjustable parameters is presented and analyzed via the three-step procedure. The scheme allows considerable flexibility in the controller parameteiization. Taking advantage of this flexibility requires the use of a priori information about the plant to be controlled. Hence, the scheme provices a

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INTEGRAL MANIFOLDS OF SLOW ADAPTATION

BY

BRADLEY DEAN RIEDLE

B.S., University of Illinois, 1982
M.S., University of Illinois, 1984

THESIS

Submitted in partial fulfillment of the requirements
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Urbana, Illinois

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Bradley Dean Riedle, Ph.D.
Department of Electrical and Computer Engineering
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CHAPTER 1

INTRODUCTION

1.1. Why Slow?

The first question which is asked when the topic of slow adaptation is introduced is "Why slow?" Here are several answers to this question.

(1) In many control systems the plant parameters change very slowly with respect to the time constants of the closed-loop system with fixed controller and plant parameters, or the plant parameters make infrequent step changes. For such systems a fixed parameter controller provides good performance for some initial interval, but performance and even stability can be lost as the plant parameters drift from the initial values. This situation is ideal for slow adaptation which can either continuously retune the controller parameters, or be turned on for finite intervals as an on-line, on-demand tuning algorithm.

(2) Using slow adaptation, adaptive control systems can be designed for given controller parametrizations. That is, the controller parametrization is chosen as necessary for the design of a good fixed parameter controller, and then, a parameter update law is designed for the given parametrization. This contrasts with the theory for fast adaptation which requires the use of a particular controller parametrization (typically ARMA) which is chosen for the ease of theoretical parameter convergence analysis.

(3) Slow adaptation replaces the exact matching conditions found in the theory for fast adaptation with a compatibility requirement that the fixed parameter controller can be tuned to give the desired performance by adjusting only the parameters. A compatible controller never requires more parameters than the exact matching conditions and usually requires many fewer parameters. The reduced number of parameters reduces the number of frequencies which inputs to the system must contain in order to be sufficiently rich.

1.2. Summary of Results

The systems we study in this monograph reduce when the parameters are constant to linear. time-invariant systems driven by inputs which are independent of the parameters. Letting x denote the states and θ denote the parameters. we study systems in the form

$$\dot{x} = A(\theta)x + B(\theta)\,w(t) \tag{1.1}$$

$$\dot{\theta} = \epsilon f(t,\theta,x) \tag{1.2}$$

for continuous time or. for discrete time. in the form

$$x(k+1) = A(\theta(k))\,x(k) + B(\theta(k))\,w(k) \tag{1.3}$$

$$\theta(k+1) = \theta(k) + \epsilon\, f(k,\theta(k),x(k))\,, \tag{1.4}$$

where x contains the states of the plant. the model. the dynamic controller. and any filters which process signals before they enter the adaptation scheme. and where w(t) is a vector input containing the reference input and any disturbances entering the system.

Slow adaptation is forced upon the system (1.1)-(1.2) by choosing ϵ small. An intuitively appealing approximation of the solutions of (1.1)-(1.2) is obtained by a two-step procedure. First assume that θ is constant in (1.1) and evaluate the solution as a function of t and θ. Assuming that $A(\theta)$ is Hurwitz. we define the frozen parameter response $\nu(t,\theta)$ by

$$\nu(t,\theta) = \int_{-\infty}^{t} e^{A(\theta)(t-s)}\,B(\theta)\,w(s)\,ds\,. \tag{1.5}$$

which is simply the response of the linear time-invariant system (1.1) to the input w(t) with initial condition zero at time $t_0 = -\infty$. The second step is to substitute $\nu(t,\theta)$ for x in (1.2). that is.

$$\dot{\theta} = \epsilon f(t,\theta,\nu(t,\theta))\,. \tag{1.6}$$

Then. letting $\tilde{\theta}(t;t_0,\theta_0)$ denote the solution of (1.6) with initial data $\theta(t_0) = \theta_0$. and letting $x(t;t_0,\theta_0,x_0)$. $\theta(t;t_0,\theta_0,x_0)$ denote the solution of (1.1)-(1.2) with initial data $x(t_0) = x_0$. $\theta(t_0) = \theta_0$. the approximation is given by

$$\theta(t;t_0,\theta_0,x_0) \cong \tilde{\theta}(t;t_0,\theta_0) \tag{1.7}$$

$$x(t;t_0,\theta_0,x_0) \cong \nu(t,\tilde{\theta}(t;t_0,\theta_0)) , \tag{1.8}$$

with the second approximation holding only after an initial transient in x. This implies that the long-term behavior of the slowly adapting system (1.1)-(1.2) is approximated by the reduced-order system (1.6). This reduced-order system is interpreted as the parameter update law without transients in x. Because the frozen parameter response $\nu(t,\theta)$ is simply the steady-state response of (1.1) with θ constant, it is easily evaluated and understood, hence, (1.6) is useful for both the analysis and design of slowly adapting systems. This approach has been used by Astrom (1983,1984) in an analysis of a specific adaptive scheme.

The idea of using the reduced-order system (1.6), which ignores the initial condition on x, as a model for the complete system (1.1)-(1.2) is similar to ignoring the boundary layer in singular perturbations, Kokotovic, Khalil, and O'Reilly, (1986). However, the presence of time-varying input w(t) in (1.1) prevents the application of the usual singular perturbation techniques for establishing the approximation (1.7)-(1.8). In Chapter 2 and in Riedle and Kokotovic (1986a), we apply integral manifold theory to the study of (1.1)-(1.2) and prove that there exists a function $g(t,\theta;\epsilon)$ with the property that if $x_0 = g(t_0,\theta_0;\epsilon)$ then $x(t;t_0,\theta_0,x_0) = g(t,\theta(t;t_0,\theta_0,x_0);\epsilon)$ for $t \geq t_0$. We also show that along solutions of (1.1)-(1.2) with $x_0 \neq g(t_0,\theta_0;\epsilon)$ the difference $x(t) - g(t,\theta(t);\epsilon)$ decays exponentially to zero. That is, for certain initial conditions or after the state transient decays, the reduced-order system

$$\dot{\theta} = \epsilon f(t,\theta,g(t,\theta;\epsilon)) \tag{1.9}$$

and the algebraic equation $x(t) = g(t,\theta(t);\epsilon)$ provide an exact description of the slow adaptation of (1.1)-(1.2). This function g defines an integral manifold M_ϵ of (1.1)-(1.2) by

$$M_\epsilon = \{t,\theta,x : x = g(t,\theta;\epsilon)\} . \tag{1.10}$$

Furthermore, the difference $h(t,\theta;\epsilon) = g(t,\theta;\epsilon) - \nu(t,\theta)$ between the function g and the frozen parameter response ν is $O(\epsilon)$. Hence, the existence of M_ϵ implies that the approximation (1.7)-(1.8)

and the approximate reduced-order system (1.6) are justified.

Although the reduced-order system (1.9) is easier to study than the complete system (1.1)-(1.2), it is still a system of nonlinear time-varying differential equations. Noting that (1.9) is in a standard Bogoliubov form for the method of averaging, the solutions of (1.9) (or (1.6)) are approximated by the solutions of the time-invariant nonlinear differential equation

$$\frac{d}{d\tau}\overline{\theta} = \overline{f}(\overline{\theta}), \tag{1.11}$$

where $\tau = \epsilon t$ is slow time and where $\overline{f}$ is the time average of f for fixed θ,

$$\overline{f}(\theta) = \lim_{T\to\infty} \frac{1}{T} \int_{t}^{t+T} f(s,\theta,\nu(s,\theta))ds. \tag{1.12}$$

Hale (1980), Meerkov (1973), Sethna and Moran (1968), Volosov (1962), and Bogoliubov and Mitropolski (1961). The method of averaging was used to simplify the analysis of (1.6) in Astrom (1983,1984).

The method of averaging gives more than a simplification of the analysis. By interpreting the stability and instability conditions provided by averaging theory in terms of the signals and transfer functions in the adaptive system, we developed a signal dependent stability criterion for slow adaptation of the Narendra and Valavani (1978) adaptive control algorithm designed for a relative degree one, order n plant but applied to a plant of order $n_p > n$ with unspecified relative degree, Riedle and Kokotovic (1985) and Kokotovic, Riedle, and Praly (1985). At that time, the integral manifold theory had not yet been applied to (1.1)-(1.2); hence, the transformation of (1.1)-(1.2) into the standard form (1.9) was not available. The stability criterion was established by linearizing the adaptive system (1.1)-(1.2) and then performing a time-varying transformation of the linearized equations into a standard form for the method of averaging. This transformation is used in several subsequent works which also obtain local results via averaging theory, Fu, Bodson, and Sastry (1985), Kosut, Anderson, and Mareels (1985), Anderson et al. (1986), and Bodson et al. (1985). After showing that the Narendra and Valavani (1978) controller possesses an

integral manifold under slow adaptation. Chapter 2 concludes with a more complete discussion of the stability criterion results and an estimate of the region of attraction which is not dependent on linearization.

In view of the similarity between the continuous-time system (1.1)-(1.2) and the discrete-time system (1.3)-(1.4), it is tempting to simply state that the discrete-time counterparts of the results of Chapter 2 hold with appropriate modifications of the proofs. However, this claim has met with some skepticism and the supporting literature for ordinary difference equations is not as extensive as that for ordinary differential equations. Therefore, we take this opportunity to present in Chapter 3 a complete self-contained proof of these results for the discrete-time slowly adapting system (1.3)-(1.4). Our proof of the existence of an integral manifold follows the proof in Chapter 2 for continuous-time except that references to Chapter VII of Hale (1980) are replaced with a complete derivation of the required bounds. Using a different proof, Praly (1986) has also shown the existence of an integral manifold of (1.3)-(1.4).

With the existence of an integral manifold M_ϵ established, it follows that the system (1.3)-(1.4) restricted to the manifold is described by $x(k) = g(k,\theta(k);\epsilon)$ and

$$\theta(k+1) = \theta(k) + \epsilon f(k,\theta(k),g(k,\theta(k);\epsilon)), \tag{1.13}$$

which is analogous to (1.9). While many results are available in the cited literature for averaging of the ordinary differential equation (1.9), very few results are available for the ordinary difference equation (1.13) with deterministic inputs. The notable exception to this rule is Meerkov (1973) who states theorems for discrete-time systems (but refers to the continuous-time proof). Taking inspiration from Meerkov's continuous-time proofs, we state and prove several basic averaging theorems relating the solution of (1.13) to the solutions of the ODE (1.11) with

$$\bar{f}(\theta) \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{i=k}^{k+N-1} f(i,\theta,\nu(i,\theta)), \tag{1.14}$$

where ν is now the frozen parameter response of (1.3). In contrast to the averaging theory for (1.13) with deterministic inputs, many articles have been written concerning the relationship

between the behavior of (1.13) and the ODE (1.11) when (1.13) represents a recursive adaptive algorithm with stochastic inputs. Ljung (1977). Kushner (1977). and Benveniste. Goursat. and Ruget (1980). to mention a few. Our proof is easily applied to a stochastic system on a sample path by sample path basis. Hence. some "with probability one" results for the system (1.3)-(1.4) can be obtained as corollaries of our basic averaging theorems. However. we use some boundedness assumptions and many triangle inequalities in our proof. Hence. our proof does not reproduce any of the weak convergence results.

Motivated by the method of sensitivity points (Kokotovic. 1973) and some early work on self-adaptive systems. Medanic and Kokotovic (1965) and Kokotovic. Medanic. Vuskovic. and Bingulac (1966). we present in Chapter 4 and in Riedle and Kokotovic (1986c) a controller parametrization with much more flexibility than the parametrizations usually seen in the adaptive control literature and a parameter update law which is designed with the intention of using slow adaptation. This parametrization allows the number of adjustable gains to be chosen independently from the order of the fixed gain feedback controller. Hence. it provides the freedom to design adaptive control systems with only a few adjustable parameters. Along with this freedom comes the necessity (and hence. opportunity) to use much *a priori* information about the plant and to make a more extensive off-line design effort. The additional design effort is compensated by improved performance and confidence in the on-line operation of the slowly adapting system.

As noted earlier. slow adaptation allows the controller parametrization to be specified for the design of a good fixed parameter controller. After the controller parametrization is specified. the design of the slowly adapting system is completed by developing a parameter update law. In Chapter 5 we illustrate the development of a parameter update law for a given controller parametrization. The plant is fifth order with three uncertain parameters and the controller is first order with three adjustable parameters. The uncertain plant parameters can vary from given nominal values by 30%. We first do off-line numerical analysis to verify that the controller can be tuned for all possible values of the plant parameters. We then present simulation results which

show that the controller parameters of the slowly adapting system converge to the values which achieve optimal tuning in the off-line numerical analysis. We conclude by showing that the theoretical analysis of this algorithm is similar to the analysis in Chapter 4 and that the positive simulation results are predicted by the analysis.

CHAPTER 2

INTEGRAL MANIFOLDS OF SLOW ADAPTATION IN CONTINUOUS TIME

2.1. Introduction

Continuous-time adaptive algorithms for estimation and control can be represented by the nonlinear dynamic system

$$\dot{x} = A(\theta)x + B(\theta)w(t) \quad x \in R^{n_x}, w \in R^{r} \tag{2.1}$$

$$\dot{\theta} = \epsilon f(t,\theta,x), \quad \theta \in R^{n_\theta}. \tag{2.2}$$

The x-equation (2.1), where $w(t)$ incorporates both the reference and disturbance inputs, describes the plant, its controller, filters, etc.; hence, the n_x-vector x is referred to as a "state." The θ-equation (2.2) is the update law for the n_θ-vector of adjustable "parameters." When x and θ strongly, interact the distinction between the "states" and "parameters" is meaningless. However, in the case of "slow adaptation" this distinction is meaningful and greatly simplifies the analysis. In the system (2.1)-(2.2) the slow adaptation is due to the smallness of the scalar gain ϵ, which forces $\dot{\theta}$ to be small and the parameters θ to evolve slowly compared to the states x. Even without this scaling by ϵ, a typical adaptive transient consists of a few rapid initial swings after which the parameters continue to move slowly as $f(t,\theta,x)$ becomes small. During the period of slow adaptation the parameters may (a) remain in a bounded set where Re $\lambda(A(\theta)) < 0$, (b) drift toward infinity with Re $\lambda(A(\theta)) < 0$, or (c) drift to a region where Re $\lambda(A(\theta)) > 0$.

In this chapter the concept of slow adaptation is made precise by showing that it occurs in an *integral manifold* M_ϵ of (2.1)-(2.2), a time-varying n_θ-dimensional surface in the $n_x + n_\theta$-dimensional space of x and θ, defined by

$$M_\epsilon = \{t,\theta,x : x = g(t,\theta;\epsilon)\} \, . \tag{2.3}$$

where $\nu(t,\theta) = g(t,\theta;0)$ is the steady-state response of (2.1) with constant θ. In Section 2.2 we show that $g(t,\theta;\epsilon)$ can be viewed as a similar steady-state response in the case of slow variations of

θ. For this reason we call M_ϵ a "slow manifold" of (2.1)-(2.2). The motion of the parameters θ in the slow manifold is governed by the update law (2.2), but with x replaced by $g(t,\theta;\epsilon)$, that is,

$$\dot{\theta} = \epsilon f(t,\theta, g(t,\theta;\epsilon)) . \tag{2.4}$$

For an adaptive system this equation is an *exact description of the adaptation process after the state transients have decayed.*

In Section 2.3 we formulate conditions for the existence of $h(t,\theta;\epsilon) = g(t,\theta;\epsilon) - \nu(t,\theta)$, and in Section 2.4 we give conditions for the slow manifold M_ϵ to be attractive, as well as a procedure for estimating the region of attraction. By showing that M_ϵ is attractive and that $h(t,\theta;\epsilon) \to 0$ as $\epsilon \to 0$, we justify the use of

$$\dot{\theta} = \epsilon f(t,\theta,\nu(t,\theta)) \tag{2.5}$$

as an approximation of (2.4) for ϵ sufficiently small. This approximation combined with an averaging analysis of (2.10) was instrumental in Astrom's (1983,1984) lucid explanation of the drift instability observed by Rohrs et al. (1982, 1985). In Section 2.5 and in Riedle and Kokotovic (1986b) we show that the given conditions for the existence of an attractive slow manifold are met by a standard model reference adaptive control system. The results of Section 2.6 prove the asymptotic validity of Astrom's approach and generalize the analysis which led to the local stability criteria formulated by Riedle and Kokotovic (1985), Kokotovic, Riedle, and Praly (1985), Kosut, Anderson, and Mareels (1985), Fu, Bodson, and Sastry (1985), and Riedle, Praly, and Kokotovic (1986).

Before we proceed, let us mention that the concept of an integral manifold was introduced by Lyapunov and used by him and Perron in their studies of conditionally stable systems. More recently this concept is encountered in the averaging literature, Bogoliubov and Mitropolski (1961), Volosov (1962), and Mitropolski and Lykova (1973). A comprehensive treatment, independent of averaging, is found in Pliss (1966, 1977) and Hale (1980). Closely related notions are center

manifolds: Fenichel (1971) and Carr (1981), and singular perturbations: Hoppensteadt (1971), Fenichel (1979), and Kokotovic, Khalil, and O'Reilly (1986).

2.2. Interpretation and Approximation of the Slow Manifold

An integral manifold M_ϵ of (2.1)-(2.2) is simply defined by the statement that if the vector x,θ is in M_ϵ at $t = t_o$, then it is in M_ϵ for all t, that is,

$$\begin{bmatrix} x(t_o) \\ \theta(t_o) \end{bmatrix} \in M_\epsilon \Rightarrow \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \in M_\epsilon \;\; \forall t . \tag{2.6}$$

If a manifold M_ϵ can be found for each value of ϵ in a segment $\epsilon \in [0,\epsilon^*]$, then we shall say that an ϵ-family of slow manifolds exists. The simplest member of this family is the "frozen parameter" manifold M_0 defined by $\epsilon = 0$ and the requirement that if $x = \nu(\tau,\theta)$ at $\tau = t$, then $x = \nu(\tau,\theta)$ for all $\tau \in R$. Noting from (2.2) that if $\epsilon = 0$, then θ is constant, we have

$$\nu(t,\theta) = e^{A(\theta)(t-\tau)}\nu(s,\theta) + \int_s^t e^{A(\theta)(t-\tau)}B(\theta)w(\tau)d\tau . \tag{2.7}$$

Assumption 2.1 : There exist a set $\Theta \subset R^{n_\theta}$ and constants $\alpha > 0$ and $K \geqslant 1$ such that

$$| e^{A(\theta)(t-s)} | \leqslant K e^{-\alpha_0 (t-s)} \tag{2.8}$$

where $|\cdot|$ is the Euclidean norm.

□

Under this assumption we let $s \to -\infty$ and obtain from (2.7)

$$\nu(t,\theta) = \int_{-\infty}^t e^{A(\theta)(t-\tau)} B(\theta)w(\tau)d\tau . \tag{2.9}$$

For a broad class of signals w(t), including almost periodic signals, this integral is well defined and shows that M_0 represents the familiar "steady-state response" of the linear time-invariant system (2.1) considered as a function of both t and θ. Although the situation is more complicated when $\epsilon > 0$ and θ is not constant, the interpretation of M as a "steady-state response" is still helpful.

Introducing the deviation of x from $\nu(t,\theta)$ as a new state variable

$$z = x - \nu(t,\theta), \tag{2.10}$$

we rewrite (2.1)-(2.2) in the form

$$\dot{z} = A(\theta)z - \epsilon\nu_\theta(t,\theta)F(t,\theta,z), \tag{2.11}$$

$$\dot{\theta} = \epsilon f(t,\theta,\nu(t,\theta) + z) \equiv \epsilon F(t,\theta,z), \tag{2.12}$$

where $\nu_\theta(t,\theta)$ is the $n_x \times n_\theta$ sensitivity matrix

$$\nu_\theta(t,\theta) = \frac{\partial \nu}{\partial \theta} = \int_{-\infty}^{t} e^{A(\theta)(t-\tau)} \frac{\partial}{\partial \theta} [A(\theta)x + B(\theta)u(\tau)]_{x=\nu(\tau,\theta)} d\tau. \tag{2.13}$$

The brackets indicate that $\frac{\partial}{\partial \theta}$ is performed with x fixed, after which $\nu(\tau,\theta)$ is substituted for x. Properties of the response $\nu(t,\theta)$ and its sensitivity $\nu_\theta(t,\theta)$ are among the crucial factors influencing the behavior of an adaptive scheme. We characterize these properties by assuming bounds on ν and ν_θ.

Assumption 2.2: There exist positive constants v, v_1, and v_2 such that for all $t \in R$ and θ, $\hat{\theta} \in \Theta$

$$|\nu(t,\theta)| \leq v, \; |\nu_\theta(t,\theta)| \leq v_1, \; |\nu_\theta(t,\theta) - \nu_\theta(t,\hat{\theta})| \leq v_2 \; |\theta - \hat{\theta}|. \tag{2.14}$$

□

Remark 2.1: A sufficient condition for Assumption 2.2 to be satisfied is that $A(\theta)$, and $B(\theta)$ have Lipschitzian derivatives and that w(t) is uniformly bounded. We make the assumption directly on ν to ν_θ to simplify expressions in this sequel.

□

In the (z,θ)-coordinates M_0 is defined by z=0. To define M for $\epsilon > 0$ we need to find a function h of t and θ parametrically dependent on ϵ such that

$$z = h(t,\theta;\epsilon) \tag{2.15}$$

satisfies (2.11)-(2.12). Let us first interpret $h(t,\theta;\epsilon)$ by constructing a sequence of "steady-state responses" $h_0(t,\theta;\epsilon)$, $h_1(t,\theta;\epsilon)$, . . . , $h_k(t,\theta;\epsilon)$,... which in Section 2.3 will be shown to converge to

$h(t,\theta;\epsilon)$. Suppose that $h_k(t,\theta;\epsilon)$ is available for all t and each $\theta \in \Theta$, and substitute it for z in (2.12). Then compute the solution $\theta_k(s;t,\theta,\epsilon)$ of the end-value problem

$$\frac{d}{ds}\,\theta_k(s) = \epsilon F(s,\theta_k(s),h_k(s,\theta_k(s;\epsilon))),\quad \theta_k(t) = \theta \in \Theta\,. \tag{2.16}$$

With $\theta_k(s) = \theta_k(s;t,\theta,\epsilon)$ and $h_k(s,\theta_k(s))$ available, use (2.11) to evaluate h_{k+1} along $\theta_k(s)$ from

$$\frac{d}{ds}\,h_{k+1}(s,\theta_k(s);\epsilon) = A(\theta_k(s))h_{k+1}(s,\theta_k(s);\epsilon) - \epsilon\nu_\theta(s,\theta_k(s))F(s,\theta_k(s),h_k(s,\theta_k(s);\epsilon))\,. \tag{2.17}$$

The state transition matrix $\Phi_k(s,\tau) = \Phi_k(s,\tau;t,\theta,\epsilon)$ of the linear time-varying system (2.17) is defined by

$$\frac{\partial}{\partial s}\,\Phi_k(s,\tau) = A(\theta_k(s))\Phi_k(s,\tau),\quad \Phi_k(\tau,\tau) = I. \tag{2.18}$$

If (2.18) is exponentially stable, that is, if as in (2.8), there exist positive constants K_1 and α_1 such that

$$|\Phi_k(s,\tau)| \leqslant K_1 e^{-\alpha_1(s-\tau)},\quad \forall s \geqslant \tau,\quad \forall \tau \in R, \tag{2.19}$$

then the steady-state response of (2.17), analogous to (2.9), is

$$h_{k+1}(s,\theta_k(s)) = -\epsilon\int_{-\infty}^{s} \Phi_k(s,\tau)\nu_\theta(\tau,\theta_k(\tau))F(\tau,\theta_k(\tau),h_k(\tau,\theta_k(\tau)))d\tau. \tag{2.20}$$

This expression defines h_{k+1} along a particular trajectory $\theta_k(s)$ whose "end"-point at s=t is θ. By choosing different "end"-points $\theta \in \Theta$, hence, different trajectories $\theta_k(s) = \theta_k(s;t,\theta,\epsilon)$, the function

$$h_{k+1}(t,\theta;\epsilon) = -\epsilon\int_{-\infty}^{t} \Phi_k(t,\tau)\nu_\theta(\tau,\theta_k(\tau))F(\tau,\theta_k(\tau),h_k(\tau,\theta_k(\tau)))d\tau \tag{2.21}$$

can be evaluated for each $\theta \in \Theta$ and all $t \in R$. Except for the use of different trajectories $\theta_k(\tau) = \theta_k(\tau;t,\theta,\epsilon)$ in place of different constant values of θ, there is a clear analogy between $h_{k+1}(t,\theta;\epsilon)$ defined by (2.21) and $\nu(t,\theta)$ defined by (2.9). Initialized with $h_0(t,\theta;\epsilon) \equiv 0$, the sequence $h_k(t,\theta;\epsilon)$, k=1,2,... is uniquely defined by (2.16) and (2.21). These expressions, which are not recommended as a computational procedure, will be used in Section 2.3 to prove the existence, and other properties, of $h(t,\theta;\epsilon)$.

If a continuously differentiable $h(t,\theta;\epsilon)$ is known to exist, then the substitution of

$$\dot{z} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial \theta}\dot{\theta} \tag{2.22}$$

into (2.11)-(2.12) shows that $h(t,\theta;\epsilon)$ satisfies the partial differential equation

$$\frac{\partial h}{\partial t} = A(\theta)h - \epsilon(\nu_\theta(t,\theta) + \frac{\partial h}{\partial \theta})F(t,\theta,h) \tag{2.23}$$

which suggests that it may be computationally feasible to approximate $h(t,\theta;\epsilon)$ by a power series in ϵ. Substituting

$$h(t,\theta;\epsilon) = h^0(t,\theta) + \epsilon h^1(t,\theta) + \epsilon^2 h^2(t,\theta) + \cdots \tag{2.24}$$

into (2.23) and evaluating the terms of the series, we see that $h^0(t,\theta) = 0$ and that $h^1(t,\theta)$ is the steady-state response of

$$\frac{\partial h^1}{\partial t} = A(\theta)h^1 - \nu_\theta(t,\theta)F(t,\theta,0). \tag{2.25}$$

The equations for h^2, h^3, etc. are more complicated and, from a practical point of view, the approximation $h(t,\theta;\epsilon) \doteqdot \epsilon h^1(t,\theta)$ may be all that is needed to improve the "frozen parameter" approximation (2.5), because $h^1(t,\theta)$ incorporates the effects of $\nu_\theta(t,\theta)$, which are important when the sensitivity of the plant with respect to adjustable parameters is high.

Example 2.1: The analysis of the effects of an unstable zero $\frac{1}{\mu} > 0$ on the performance of an adaptive controller designed for a minimum phase plant is nontrivial even in the case of a first order plant and a single adjustable parameter. Such an adaptive system, shown in Fig. 2.1, is described by

$$\dot{x} = \frac{1-\theta}{1-\mu\theta}x + \frac{1-\mu}{1-\mu\theta}r \tag{2.26}$$

$$\dot{\theta} = \epsilon\frac{x-\mu r}{1-\mu\theta}\left(\frac{x-\mu r}{1-\mu\theta} - y_m\right) \tag{2.27}$$

where $r = r(t)$ and $y_m = y_m(t)$ are, respectively, the reference input and model output. For $r = \cos\omega t$ the frozen parameter response $\nu(t,\theta)$ and its sensitivity are

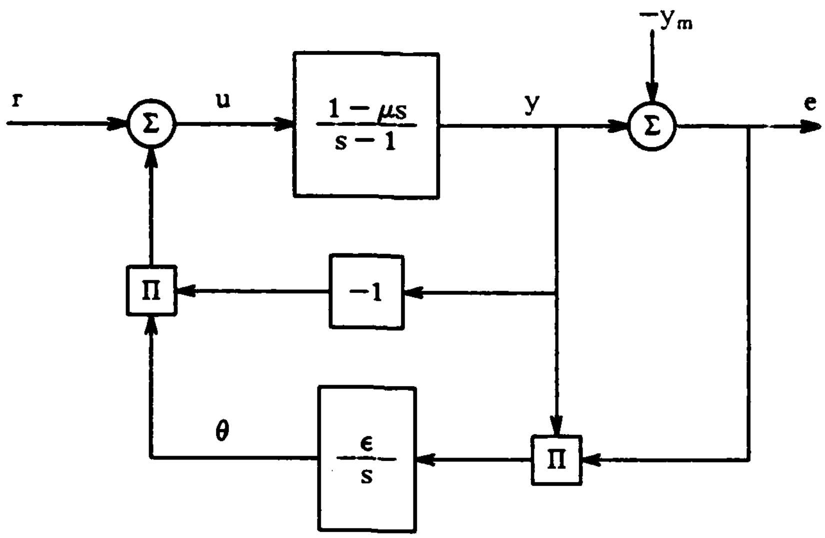

Fig. 2.1. Block diagram of the system (2.26)-(2.27).

$$\nu(t,\theta) = V(\theta,\omega)\cos(\omega t + \phi(\theta,\omega)) \tag{2.28}$$

$$V(\theta,\omega) = \frac{1-\mu}{\sqrt{(1-\mu\theta)^2\,\omega^2 + (\theta-1)^2}}, \quad \phi(\theta,\omega) = \arctan\left|\frac{(1-\mu\theta)\omega}{\theta-1}\right| \tag{2.29}$$

$$\nu_\theta(t,\theta) = \frac{\partial V}{\partial\theta}\cos(\omega t + \phi) - \frac{\partial\phi}{\partial\theta} V \sin(\omega t + \phi). \tag{2.30}$$

With ν and ν_θ known, h^1 can be obtained as the steady-state response of (2.25). Then $\nu(t,\theta) + \epsilon h^1(t,\theta)$ can serve as an approximation of $h(t,\theta;\epsilon)$, which, in this case, is periodic in t.

For a clear graphic display let us consider the constant input case with $r = 1$, $y_m = .5$. A simple calculation gives

$$\nu(\theta) = \frac{1-\mu}{\theta-1}, \quad \nu_\theta(\theta) = -\frac{1-\mu}{(\theta-1)^2} \tag{2.31}$$

and, upon the substitution into (2.24), we obtain

$$h^1 = \frac{1}{2(\theta-1)^5}(1-\mu)(1-\mu\theta)(3-\theta). \tag{2.32}$$

For $\theta \geqslant 2.25$ the sensitivity ν_θ is small and, as predicted, the slow manifold is practically indistinguishable from $\nu(\theta)$. For $\theta \geqslant 2.25$ the trajectories plotted for $\mu = .25$ and $\epsilon = .1$ in Fig. 2.2

clearly show a separation of time scales: the slope $\frac{dx}{d\theta}$ is much steeper away from the manifold than along the manifold. Again as predicted, the situation changes in the region $\theta \leqslant 2$ where the sensitivity ν_θ is high. In this region, the curve $\nu(\theta) + \epsilon h^1(\theta)$ is a significantly better approximation of the slow manifold than $\nu(\theta)$. The disastrous effect of the unstable zero $\frac{1}{\mu}$ is also characteristic: for $\theta > \frac{1}{\mu}$ the manifold is repulsive, whereas for $\theta < \frac{1}{\mu}$, it is attractive. For $\theta = \frac{1}{\mu}$ the system (2.26)-(2.27) is not defined. In the manifold, the slow adaptation converges to the equilibrium $\theta = 3$.

Remark 2.2: To avoid excessive numerical sensitivity of the unstable trajectories for $\theta > 4$, they have been obtained by simulation in reverse time.

□

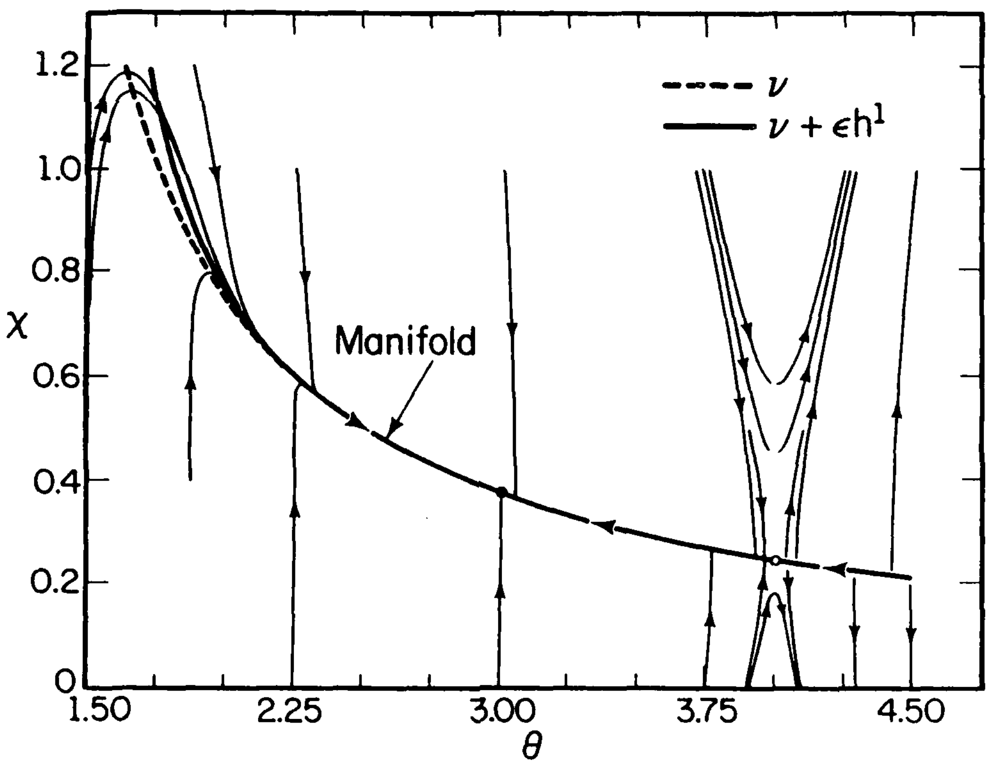

Fig. 2.2. Trajectories of (2.26)-(2.27) with $r = 1$, $y_m = 0.5$, $\epsilon = 0.1$, $\mu = 0.25$ converge to the manifold for $\theta < 4$. In the manifold all movement is toward $\theta = 3$.

Remark 2.3: An analytical study of repulsive manifolds would involve the following modifications. If $\mathrm{Re}\,\lambda_i(A(\theta)) > 0$ for all $i = 1,\dots,m$, then (2.11),(2.13), and (2.20) are to be integrated in reverse time, from ∞ to t. If A has both stable and unstable eigenvalues, then each of these expressions would include two integrals, one from $-\infty$ to t for the stable part and one from ∞ to t for the unstable part of the response. We restrict our analysis to attractive slow manifolds.
□

2.3. Existence of the Slow Manifold

Expressions (2.16) and (2.21) rewritten as $h_{k+1} = Th_k$ define a map T. Its fixed point, if it exists, is our function $h(t,\theta;\epsilon)$. As in any fixed point argument, we first specify a closed subset of a Banach space in which to search for $h(t,\theta;\epsilon)$. We let this space be the set of continuous functions $H(t,\theta)$ equipped with the norm $\|H\| = \sup_{t,\theta \in R \times R^{n_\theta}} |H(t,\theta)|$ and use positive constants D and Δ to define our closed subset $\mathbf{H}(D,\Delta)$ as

$$\mathbf{H}(D,\Delta) = \{H: R\times R^n \to R^m \mid \|H\| \leqslant D \text{ and } |H(t,\theta) - H(t,\hat{\theta})| \leqslant \Delta|\theta - \hat{\theta}|,\ \forall t \in R;\ \theta,\hat{\theta} \in R^{n_\theta}\}. \tag{2.33}$$

Our goal is not only to establish the existence of $h(t,\theta;\epsilon)$, but also to characterize it by estimating D and Δ in terms of ϵ and the data about the adaptive system. In addition to Assumptions 2.1 and 2.2 about the system (2.1) with constant θ, we need the following data about the parameter update law (2.12).

Assumption 2.3 : There exist nondecreasing positive functions $\rho_F(D)$, $\rho_\theta(D)$, and $\rho_z(D)$ such that

$$\begin{aligned} &|F(t,\theta,z)| \leqslant \rho_F(D), \quad |F(t,\theta,z) - F(t,\hat{\theta},z)| \leqslant \rho_\theta(D)\,|\theta - \hat{\theta}|, \\ &|F(t,\theta,z) - F(t,\theta,\hat{z})| \leqslant \rho_z(D)|z - \hat{z}|, \quad \forall t \in R;\ \ \theta,\hat{\theta} \in \Theta;\ \ |z|,|\hat{z}| \leqslant D. \end{aligned} \tag{2.34}$$

□

Remark 2.4: A sufficient condition for Assumption 2.3 to hold is that $f(t,\theta,x)$ be bounded, and Lipschitzian in θ,x uniformly with respect to $t \in R$, $\theta \in \Theta$ in x in compact sets. These are very mild conditions which are met by most parameter update laws.

□

In the iterative scheme (2.16), (2.21) the stability condition (2.19) is crucial. Unfortunately, our Assumption 2.1 is not sufficient to guarantee that each trajectory $\theta_k(s;t,\theta,\epsilon)$ will remain in the set Θ for all s, as required by (2.19). Following Carr (1981), we avoid this difficulty by proving the existence of $h(t,\theta;\epsilon)$ for a modified system, rather than for (2.11)-(2.12). In the modified system, each θ which is not in Θ is replaced by some value $p(\theta)$ which remains in Θ. By construction, the original and the modified systems are identical for all $\theta \in \Theta$, that is, $p(\theta) = \theta$, $\forall\theta \in \Theta$. For simplicity, we restrict the set Θ in Assumption 2.1 to be convex and compact. Then $p(\theta)$ is uniquely defined as the point in Θ closest to θ, namely,

$$p(\theta) = \arg \min_{p \in \Theta} |p - \theta|. \tag{2.35}$$

We henceforth analyze the modified system

$$\dot{z} = A(p(\theta)) - \epsilon\nu_\theta(t,p(\theta))\, F(t,p(\theta),z), \tag{2.36}$$

$$\dot{\theta} = \epsilon F(t,\theta,p(\theta),z). \tag{2.37}$$

The results obtained for the modified system translate into those for the original system as follows. Suppose that (2.36)-(2.37) has an integral manifold $h(t,\theta;\epsilon)$ and a solution $\theta(t)$ of (2.37) with $z = h(t,\theta;\epsilon)$ which satisfies $\theta(t) \in \Theta$, $\forall t \in [t_0,t_1]$. Then $\theta(t)$, $z = h(t,\theta(t);\epsilon)$ is also a solution of the original system (2.11)-(2.12) $\forall t \in [t_0,t_1]$. As for the modified assumptions, we note that, because $p(\theta) \in \Theta$ and $|p(\theta) - p(\hat{\theta})| \leq |\theta - \hat{\theta}|$, the bounds imposed on $A(\theta)$, $\nu(\theta)$, $\nu_\theta(t,\theta)$, $F(t,\theta,z)$ for all $\theta \in \Theta$ are satisfied by $A(p(\theta))$, $\nu(t,p(\theta))$, $\nu_\theta(t,p(\theta))$, $F(t,p(\theta),z)$ for all $\theta \in R^{n_\theta}$. To describe the dependence of $F(t,p(\theta),z)$ and $\nu_\theta(t,p(\theta))\, F(t,p(\theta),z)$ on θ over the set $\mathbf{H}(D,\Delta)$ we define $\rho_1(D,\Delta)$ and $\rho_2(D,\Delta)$ such that

$$|F(t,p(\theta),H(t,\theta)) - F(t,p(\hat{\theta}),H(t,\hat{\theta}))| \leq \rho_1(D,\Delta)\,|\theta - \hat{\theta}| \tag{2.38}$$

$$|\nu_\theta(t,p(\theta))F(t,p(\theta),H(t,\theta)) - \nu_\theta(t,p(\hat{\theta}))F(t,p(\hat{\theta}),H(t,\hat{\theta}))| \leq \rho_2(D,\Delta)\,|\theta - \hat{\theta}| \tag{2.39}$$

for all $t \in R$, $\theta,\hat{\theta} \in R^{n_\theta}$ and $H \in \mathbf{H}(D,\Delta)$. It follows from Assumptions 2.2 and 2.3 that ρ_1 and ρ_2 exist and can be chosen to satisfy

$$\rho_1(D,\Delta) \leqslant \rho_\theta(D) + \Delta\rho_z(D), \quad \rho_2(D,\Delta) \leqslant v_1\rho_1(D,\Delta) + v_2\rho_F(D). \tag{2.40}$$

We now perform the same modification of the iterative expressions (2.16) through (2.21). In particular, (2.16) becomes

$$\frac{d\theta_k(s)}{ds} = \epsilon F(s,p(\theta_k(s)),\, h_k(s,\theta_k(s);\epsilon)), \quad \theta_k(t) = \theta. \tag{2.41}$$

As before, the trajectory $\theta_k(s;t,\theta,\epsilon)$ is determined by its "end"-condition θ at s=t. However, the modification now guarantees that to each $h_k \in \mathbf{H}(D,\Delta)$ and each $\theta \in R^{n_\theta}$ there corresponds a unique continuous solution of (2.41) $\theta_k(s;t,\theta,\epsilon) = \theta_k(s)$, defined for all $s \in R$. This is a consequence of the global character of (2.38). A more important advantage of the modification is that the stability property (2.19) of $\Phi_k(s,\tau;t,\theta,\epsilon)$, the solution

$$\frac{\partial}{\partial s}\,\Phi_k(s,\tau) = A(p(\theta_k(s)))\,\Phi_k(s,\tau), \quad \Phi(\tau,\tau) = I, \tag{2.42}$$

can be established as follows.

Lemma 2.1 : Suppose that the Assumptions 2.1, 2.2, and 2.3 hold; choose a constant $a > 0$ such that $|A(\theta) - A(\hat\theta)| \leqslant a|\theta - \hat\theta|$ for all $\theta,\hat\theta \in \Theta$ and let

$$\rho_3(D) = [a\rho_F(D)\,K\,\ln K]^{1/2}. \tag{2.43}$$

If $h_k \in \mathbf{H}(D,\Delta)$ and

$$\epsilon a\rho_F(D) < \alpha_o^2(K\,\ln K)^{-1} \tag{2.44}$$

then

$$|\Phi_k(s,\tau)| \leqslant K e^{-\alpha_1(\epsilon,D)(s-\tau)}, \quad \forall s \geqslant \tau \tag{2.45}$$

where $\alpha_1(\epsilon,D) = \alpha_o - \epsilon^{1/2}\rho_3(D)$.

Proof : By assumption, for all $s,\hat s \in R$,

$$|A(p(\theta_k(s))) - A(p(\theta_k(\hat s)))| \leqslant a|\theta_k(s) - \theta_k(\hat s)| \leqslant \epsilon a\rho_F(D)|s - \hat s|, \tag{2.46}$$

and by construction, $A(p(\theta))$ satisfies (2.8) for all $\theta \in R^{n_\theta}$. The proof then follows from a standard

theorem for systems with slowly varying coefficients, e.g., page 117 of Coppel (1965) or Section 2.5 of Kokotovic, Khalil, and O'Reilly (1986).

□

We are now prepared to consider the map T defined pointwise via

$$(Th_k)(t,\theta;\epsilon) = -\epsilon \int_{-\infty}^{t} \Phi_k(t,\tau)\, \nu_\theta(\tau,p(\theta_k(\tau)))\, F(\tau,p(\theta_k(\tau)),h_k(\tau,\theta_k(\tau);\epsilon))\, d\tau\ , \tag{2.47}$$

where $\theta_k(\tau) = \theta_k(\tau;t,\theta,\epsilon)$ and $\Phi_k(t,\tau) = \Phi_k(t,\tau;t,\theta,\epsilon)$. The meaning of T is made clear by comparing (2.47) with (2.21), that is, the map T represents the iterations (2.16) through (2.21) for the modified system (2.36)-(2.37).

Lemma 2.2 : Suppose that Assumptions 2.1, 2.2, and 2.3 hold. If ϵ, D, and Δ satisfy (2.44) and

$$\epsilon K v_1 \rho_F(D) / \alpha_1(\epsilon,D) \leqslant D \tag{2.48}$$

$$\epsilon \rho_1(D,\Delta) < \alpha_1(\epsilon D) \tag{2.49}$$

$$\epsilon \frac{K}{\alpha_1(\epsilon,D) - \epsilon\, \rho_1(D,\Delta)} \left| \rho_2(D,\Delta) + \frac{K a v_1 \rho_F(D)}{\alpha_1(\epsilon,D)} \right| \leqslant \Delta \tag{2.50}$$

$$\epsilon \frac{\rho_z(D)}{\alpha_1(\epsilon,D)} [K v_1 + \Delta] < 1 \tag{2.51}$$

then T is a contraction mapping on $\mathbf{H}(D,\Delta)$.

Proof(discussion): Omitting lengthy calculations of the bounds (2.48)-(2.51), we only indicate their origin and discuss their meaning. Using (2.45) and (2.47) it is not difficult to see that (2.48) assures $\| Th_k \| \leqslant D$. The most complicated bound (2.50) originates from $(Th_k)(t,\theta;\epsilon) - (Th_k)(t,\hat{\theta};\epsilon)$ written as the sum of two integrals

$$\begin{aligned} &-\epsilon \int_{-\infty}^{t} \Phi_k(t,\tau)[\nu_\theta(\tau,p(\theta_k(\tau)))F(\tau,p(\theta_k(\tau)),h_k(\tau,\theta_k(\tau))) \\ &\qquad - \nu_\theta(\tau,p(\hat{\theta}_k(\tau)))F(\tau,p(\hat{\theta}_k(\tau)),h_k(\tau,\hat{\theta}_k(\tau)))]d\tau \\ &- \epsilon \int_{-\infty}^{t} [\Phi_k(t,\tau) - \hat{\Phi}_k(t,\tau)]\nu_\theta(\tau,p(\hat{\theta}_k(\tau)))F(\tau,p(\hat{\theta}_k(\tau)),h_k(\tau,\hat{\theta}_k(\tau)))d\tau \end{aligned} \tag{2.52}$$

where $\hat{\theta}_k(\tau) = \theta_k(\tau;t,\hat{\theta},\epsilon)$ and $\hat{\Phi}_k(t,\tau) = \Phi_k(t,\tau;t,\hat{\theta},\epsilon)$. Now (2.49) assures that the norm of the first integral is bounded by

$$|\theta - \hat{\theta}|\, K\rho_2(D,\Delta)[\alpha_1(\epsilon,D) - \epsilon\rho_1(D,\Delta)]^{-1} \quad . \tag{2.53}$$

hence, is well defined over any infinite interval $(-\infty,t]$. The term in the brackets, also appearing in (2.49) is of conceptual interest, because it represents a time-scale separation property. To see this, note from (2.38) and (2.41) that $|\theta_k(s) - \hat{\theta}_k(s)| \leqslant |\theta - \hat{\theta}|\, e^{\epsilon\rho_1(D,\Delta)|s-t|}$; hence, $\epsilon\rho_1(D,\Delta)$ is the fastest exponential rate of the "steady-state" solutions. On the other hand, (2.45) shows that $\alpha_1(\epsilon,D)$ is the slowest exponential decay towards a "steady-state" solution. If the difference of these two rates is larger, the dependence of h on θ will be "smoother." The other term in (2.50) indicates that the smallness of the sensitivity bound v_1 also contributes to the "smoothness" of h. Finally, (2.51) is a "contraction" bound for $\| Th_k - Th_m \| / \| h_k - h_m \|$. For further details in this continuous-time case see Chapter VII of Hale (1980). We give a complete proof following Hale for the discrete-time case in Chapter 3. To conclude, let us mention that the time-scale/smoothness relationship is clarified in Fenichel (1971).

□

Remark 2.5 : The only use of (2.44) in Lemma 2.2 is to ensure (2.45) holds with $\alpha_1(\epsilon,D) > 0$. If (2.45) can be established for $\alpha_1(\epsilon,D) > 0$ without (2.44), then (2.44) can be dropped as a hypothesis of Lemma 2.2. For example, if there exists a constant positive definite symmetric matrix P which satisfies

$$A^T(\theta)P + PA(\theta) \leqslant -\alpha_0 P \qquad \forall\theta \in \Theta \, . \tag{2.54}$$

then (2.45) is satisfied with $\alpha_1(\epsilon,D) = \alpha_0$ and $K = (\lambda_{max}P/\lambda_{min}P)^{1/2}$.

□

It is clear that for any positive D_0, $\alpha_1(\epsilon,D_0)$ will be positive for ϵ sufficiently small. With this observation in mind, it is obvious that (2.45)-(2.51) will be satisfied for any positive D_0, Δ_0 by a sufficiently small $\epsilon_0 > 0$, hence, for all $\epsilon \in [0,\epsilon_0]$. In view of the fact that ρ_F and ρ_2 are nondecreasing functions of D, it is clear that for $\epsilon < \epsilon_0$ we can use $D = (\epsilon/\epsilon_0)\, D_0$ and $\Delta = (\epsilon/\epsilon_0)\, \Delta_0$ instead of D_0 and Δ_0 in the definition of **H**. Hence, under the conditions of Lemma 2.2 the function

$h(t,\theta;\epsilon)$ exists and is an $O(\epsilon)$ quantity. This observation leads to the following summary of our existence results.

Theorem 2.1: Suppose that Assumptions 2.1, 2.2, and 2.3 hold. Then, given any fixed $D_0 > 0$ and $\Delta_0 > 0$, there exists $\epsilon_0 > 0$ such that for each $\epsilon \in [0,\epsilon_0]$ the modified system (2.36)-(2.37) has an integral manifold uniquely defined by

$$M_\epsilon = \{t,\theta,z : z = h(t,\theta;\epsilon)\}, \quad h \in \mathbf{H}\,((\epsilon/\epsilon_0)\, D_0\, .\, (\epsilon/\epsilon_0)\, \Delta_0). \tag{2.55}$$

□

When translated to the original system, this result establishes the existence of an ϵ-family of slow manifolds of (2.1)-(2.2). Recalling that $x = z + \nu(t,\theta)$, that $g = \nu + h$, and that in x,θ coordinates $M_\epsilon = \{t,\theta,x : x = g(t,\theta;\epsilon)\}$ we translate Theorem 2.1 to the original system in the following corollary.

Corollary 2.1 : Suppose that Assumptions 2.1, 2.2, and 2.3 hold. Let $x(t)$, $\theta(t)$ be the solution of (2.1)-(2.2) with initial data $\theta(t_0) = \theta_0 \in \Theta$ and $x(t_0) = g(t_0,\theta_0;\epsilon) = \nu(t_0,\theta_0) + h(t_0,\theta_0;\epsilon)$. Suppose that $\theta(t) \in \Theta$ for all $t \in [t_0,t_1]$. Given any fixed $D_0 > 0$ and $\Delta_0 > 0$, there exists $\epsilon_0 > 0$ such that for each $\epsilon \in [0,\epsilon_0]$ the solution $x(t)$, $\theta(t)$ of (2.1)-(2.2) satisfies

$$(t,\theta(t),x(t)) \in M_\epsilon = \{t,\theta,x : x = g(t,\theta;\epsilon) = \nu(t,\theta) + h(t,\theta;\epsilon)\} \tag{2.56}$$

for all $t \in [t_0,t_1]$ with $h \in \mathbf{H}((\epsilon/\epsilon_0)D_0,(\epsilon/\epsilon_0)\Delta_0)$.

□

Remark 2.6: Reference and disturbance signals are not required to be persistently exciting (PE).

□

Remark 2.7: It can be shown that if $w(t)$ and $f(t,\theta,x)$ are periodic (almost periodic) in t, then $h(t,\theta;\epsilon)$ is periodic (almost periodic) in t.

□

Example 2.2: Returning to Example 2.1, we now use Corollary 2.1 to prove that on the segment $\Theta = [2.25,3.75]$ the adaptive system (2.26)-(2.27) possesses an ϵ-family of local slow manifolds with $h \in \mathbf{H}(0.045\epsilon,4.61\epsilon)$, $\forall\epsilon \in [0,0.2]$. The Δ estimate 4.61ϵ is conservative due to the fact that $\alpha_0 = 2.85$ is evaluated at $\theta = 2.25$, whereas $a = 192$ is evaluated at $\theta = 3.75$. Less conservative estimates would result from a smaller segment Θ. In this case Lemma 2.1 is trivially satisfied and

$K = 1$ implies $\rho_3(D) = 0$. The Assumptions 2.2 and 2.3 are satisfied by

$$v = 0.6, \quad v_1 = 0.48, \quad v_2 = 0.77,$$

$$\rho_F(D) = 0.242 + 2.52D + 5.23D^2, \quad \rho_z(D) = 3.64 + 512D^2,$$

$$\rho_\theta(D) = 0.704 + 4.37D + 5.98D^2,$$

and $\rho_1(D,\Delta)$ and $\rho_2(D,\Delta)$ are taken to be

$$\rho_1(D,\Delta) = 0.704 + 4.37D + 5.98D^2 + (2.52 + 10.5D)\Delta,$$

$$\rho_2(D,\Delta) = 0.524 + 4.04D + 6.89D^2 + (1.21 + 5.04D)\Delta.$$

These expressions are valid for both cases, constant input $r=1$ and periodic input $r = \cos \omega t$, and are used to show that we can take $D_0 = .009$, $\Delta_0 = 0.922$ and $\epsilon_0 = 0.2$ in Corollary 2.1.

□

Before considering in the next section the attractivity of M_ϵ and sufficient conditions for the stability or boundedness of solutions of (2.36)-(2.37), we give an instability result. The evolution of (2.36)-(2.37) restricted to M_ϵ is governed by the reduced-order system

$$\dot{\theta} = \epsilon F(t,p,(\theta),h(t,\theta;\epsilon)) . \tag{2.57}$$

The next theorem follows from the definitions of integral manifold and instability (in the sense of Lyapunov).

Theorem 2.2 : Suppose that the conclusion of Theorem 2.1 holds and that $\epsilon \in (0,\epsilon_0]$. Let $\theta^*(t)$ be any solution of (2.57) which is bounded for finite time. If $\theta^*(t)$ is an unstable solution of (2.57), then $z^*(t) = h(t,\theta^*(t);\epsilon)$, $\theta^*(t)$ is an unstable solution of (2.36)-(2.37).

Proof : Because θ^* is an unstable solution of (2.57) there exists a $\rho > 0$ and $t_0 \in R$ such that for each $\delta > 0$ there exists a solution $\theta_\delta(t)$ of (2.57) with $|\theta^*(t_0) - \theta_\delta(t_0)| < \delta$ and $|\theta^*(t_\delta) - \theta_\delta(t_\delta)| \geq \rho$ for some $t_\delta \geq t_0$. For the same $\rho > 0$ and $t_0 \in R$ and each $\delta > 0$, the solutions $X_{\delta/(1+\Delta)}(t) = \begin{bmatrix} h(t,\theta_{\delta/(1+\Delta)}(t);\epsilon) \\ \theta_{\delta/(1+\Delta)}(t) \end{bmatrix}$ and $X^*(t) = \begin{bmatrix} z^*(t) \\ \theta^*(t) \end{bmatrix}$ of (2.36)-(2.37) satisfy

$$|X_{\delta/(1+\Delta)}(t_0) - X^*(t_0)| < \delta , \quad |X_{\Delta/(1+\Delta)}(t_{\Delta/(1+\Delta)}) - X^*(t_{\Delta/(1+\Delta)})| \geq \rho . \tag{2.58}$$

where the second inequality follows from $|[z^T\ \theta^T]| \geq |\theta|$.

□

Letting $\mathbf{B}(\delta,\hat{\theta})$ denote the ball of radius δ centered at $\hat{\theta}$,

$$\mathbf{B}(\delta,\hat{\theta}) = \{\theta : |\theta - \hat{\theta}| < \delta\},$$

we translate this instability result to the original system.

Corollary 2.2 : Suppose that the hypotheses of Theorem 2.2 are satisfied and that $\mathbf{B}(2\rho,\theta^*(t)) \subset \Theta$ for all $t \geq t_0$. Then, for $t \geq t_0$, $x^*(t) = g(t,\theta^*(t);\epsilon)$, $\theta^*(t)$ is an unstable solution of (2.1)-(2.2).

□

2.4. Attractivity of the Slow Manifold

While the existence of an integral manifold is sufficient to show that unstable solutions of (2.57) combined with $z = h(t,\theta;\epsilon)$ provide unstable solutions of (2.36)-(2.37), existence alone is not sufficient to show that stable solutions of (2.57) lead to stable solutions of (2.36)-(2.37). In this section we derive conditions under which M_ϵ is exponentially attractive and give an estimate of the region of attraction. We then give examples of how this result is used to prove that the stability properties of a solution of (2.57) are also the stability properties of the corresponding solution of (2.36)-(2.37).

Lemma 2.3 : Suppose that Assumptions 2.1, 2.2, and 2.3 hold and let ϵ, D, and Δ be such that $M = \{t,\theta,z : z = h(t,\theta;\epsilon)\}$ with $h \in \mathbf{H}(D,\Delta)$ is an integral manifold of (2.36)-(2.37). Then for every $D_1 > (K+1)D$ such that

$$\epsilon K(v_1 + \Delta)\rho_z(D_1) < \alpha_1(\epsilon,D_1) \tag{2.59}$$

the solutions $z(t)$, $\theta(t)$ of the modified system (2.36)-(2.37) starting from any bounded $\theta(t_0) = \theta_0 \in R^{n_\theta}$ and any $z(t_0) = z_0$ bounded by

$$|z_0| \leq \frac{D_1 - D}{K} - D \tag{2.60}$$

satisfy

$$|z(t) - h(t,\theta(t);\epsilon)| \leqslant K|z_0 - h(t_0,\theta_0;\epsilon)|\, e^{-\alpha_2(\epsilon,D_1)(t-t_0)} \qquad (2.61)$$

for all $t \geqslant t_0$ and any $t_0 \in R$, where $\alpha_2(\epsilon,D_1) = \alpha_1(\epsilon,D_1) - \epsilon K(v_1 + \Delta)\rho_z(D_1)$.

Proof: Suppose that $|z(t)| \leqslant D_1$ for all $t \geqslant t_0$. By Lemma 2.1

$$\dot{\hat{z}} = A(\theta(t))\hat{z} \qquad (2.62)$$

is exponentially stable with constant K and rate $\alpha_1(\epsilon,D_1)$. A converse Lyapunov theorem from Yoshizawa (1966, p. 90) shows that there exists a Lyapunov function $V(t,\hat{z})$ satisfying

$$|\hat{z}| \leqslant V(t,\hat{z}) \leqslant K|\hat{z}|, \quad |V(t,\hat{z}) - V(t,\tilde{z})| \leqslant K|\hat{z} - \tilde{z}| \qquad (2.63)$$

$$V'_{(2.36)}(t,\hat{z}) \leqslant -\alpha_1(\epsilon,D_1)\, V(t,\hat{z}), \quad t \geqslant t_0\,, \qquad (2.64)$$

where $V'_{(2.36)}$ is the upper right derivative of V along solutions of (2.36). Because $V(t,z)$ is Lipschitzian in z, h is Lipschitzian in θ, and $\theta(t)$ is Lipschitzian in t, $V(t,z - h(t,\theta;\epsilon))$ is a continuous function of t along the solutions of (2.36)-(2.37). In order to evaluate $V'_{(2.36)-(2.37)}(t,z - h(t,\theta;\epsilon))$ it is helpful to first determine expressions for $z(t+\Delta)$ given $z(t) = z$ and $h(t + \Delta,\theta(t+\Delta);\epsilon)$ given $\theta(t) = \theta$:

$$z(t+\Delta) = z + \Delta[A(p(\theta))z - \epsilon\nu_\theta(t,p(\theta))F(t,p(\theta),z))] + O(\Delta^2)\,, \qquad (2.65)$$

$$\begin{aligned} h(t+\Delta,\theta(t+\Delta);\epsilon) &= h(t+\Delta,\theta + \Delta\epsilon F(t,p(\theta),h(t,\theta;\epsilon));\epsilon) \\ &\quad + h(t+\Delta,\theta(t+\Delta);\epsilon) - h(t+\Delta,\theta + \Delta\epsilon F(t,p(\theta),h(t,\theta;\epsilon));\epsilon) \\ &= h(t,\theta;\epsilon) + \Delta[A(p(\theta)))h(t,\theta;\epsilon) - \epsilon\, \nu_\theta(t,p(\theta))F(t,p(\theta)),h(t,\theta;\epsilon))] \qquad (2.66) \\ &\quad + [h(t+\Delta,\theta + \Delta\epsilon F(t,\theta,z);\epsilon) - h(t+\Delta,\theta + \epsilon F(t,\theta,h(t,\theta;\epsilon);\epsilon)] \\ &\quad + O(\Delta^2)\,. \end{aligned}$$

From these expressions and (2.63)-(2.64) it follows that $V(t,z - h(t,\theta;\epsilon))$ satisfies

$$
\begin{aligned}
V'_{(2.36)-(2.37)}(t,z-h(t,\theta;\epsilon)) &\leqslant -\alpha_1(\epsilon,D_1)V(t,z-h(t,\theta;\epsilon)) \\
&\quad + K\,\epsilon(v_1+\Delta)\,|\,F(t,p(\theta)),z)-F(t,p(\theta)),h(t,\theta;\epsilon))\,| \\
&\leqslant -\alpha_1(\epsilon,D_1)V(t,z-h(t,\theta;\epsilon)) \\
&\quad + \epsilon\,K(v_1+\Delta)\rho_z(D_1)\,|\,z-h(t,\theta;\epsilon)\,| \\
&\leqslant -\alpha_2(\epsilon,D_1)V(t,z-h(t,\theta;\epsilon))
\end{aligned}
\tag{2.67}
$$

for all $t \geqslant t_0$, which in view of (2.63) proves (2.61) for all $t \geqslant t_0$. This argument, conditioned on the assumption that $|z(t)| \leqslant D_1$ for all $t \geqslant t_0$, also proves that (2.61) holds for $t \in [t_0,t_1]$ if $|z(t)| \leqslant D_1$ on this interval. The proof that $t_1 = \infty$ is by contradiction . Assume that there exists $t_* \in [t_0,\infty)$ such that $|z(t_*)| \geqslant D_1$ and let t_1 be the smallest such time. Since $|z(t_0)| < D_1$, $t_1 > t_0$ and

$$
|z(t_1)| \leqslant |h(t_1,\theta(t_1);\epsilon)| + |z(t_1)-h(t_1,\theta(t_1);\epsilon)| < D + K|z_0 - h(t_0,\theta_0)| \leqslant D_1, \tag{2.68}
$$

which contradicts $|z(t_1)| \geqslant D_1$.

□

Remark 2.8: If ϵ and D_1 satisfy

$$
\epsilon\, K v_1 \rho_F(D_1)/\alpha_1(\epsilon,D_1) < D_1 . \tag{2.69}
$$

then (2.60) can be relaxed to $K\,|z_0| \leqslant D_1$ when $K > 1$ or $|z_0| < D_1$ when $K = 1$. See Lemma 3.5 in Chapter 3.

□

With this remark in mind we summarize the existence and attractivity of M_ϵ in the following theorem.

Theorem 2.3: Suppose that Assumptions 2.1, 2.2, and 2.3 hold. Given any fixed $\Delta_0 > 0$, $D_0 > 0$, $D_1 > D_0$, and $\alpha \in (0,\alpha_0)$, there exists $\epsilon_1 > 0$ such that for each $\epsilon \in [0,\epsilon_1]$ there exists $h \in \mathbf{H}((\epsilon/\epsilon_1)D_0,(\epsilon/\epsilon_1)\Delta_0)$ such that $M_\epsilon = \{t,\theta,z : z = h(t,\theta;\epsilon)\}$ is an integral manifold of (2.36)-(2.37), and furthermore, solutions $z(t)$, $\theta(t)$ of (2.36)-(2.37) with $z(t_0) = z_0$ and $K\,|z_0| < D_1$ satisfy (2.61) with $\alpha_2(\epsilon,D_1) \geqslant \alpha$.

Proof: Choose ϵ_0 as for Theorem 2.1. Then choose $\epsilon_1 \leqslant \epsilon_0$ such that $\alpha_2(\epsilon_1,D_1) \geqslant \alpha$. Thus, ϵ_1 is

chosen to satisfy the most restrictive of five inequalities; hence, it is easily computed given the functions ρ.

□

Similarly to Theorem 2.1 this result can be translated to the original system (2.1)-(2.2).

Corollary 2.3: Suppose that Assumptions 2.1, 2.2, and 2.3 hold. Let $x(t)$, $\theta(t)$ be the solution of (2.1)-(2.2) with initial data $x(t_0) = x_0$, $\theta(t_0) = \theta_0 \in \Theta$. Suppose that $\theta(t) \in \Theta$ for all $t \in [t_0,t_1]$. Given any fixed $\Delta_0 > 0$, $D_0 > 0$, $D_1 > D_0$, and $\alpha \in (0,\alpha_0)$, there exists $\epsilon_1 > 0$ such that for each $\epsilon \in [0,\epsilon_1]$ there exists $h \in \mathbf{H}((\epsilon/\epsilon_1)D_0\,,(\epsilon/\epsilon_1)\Delta_0)$ with the following properties:

i) if $(t_0,\theta_0,x_0) \in M_\epsilon$, then $(t,\theta(t),x(t)) \in M_\epsilon$ for all $t \in [t_0,t_1]$.

ii) if $K\,|\,x_0-\nu(t_0,\theta_0)\,| < D_1$, then for all $t \in [t_0,t_1]$

$$|\,x(t) - g(t,\theta(t);\epsilon)\,| \leq Ke^{-\alpha(t-t_0)}\,|\,x_0 - g(t_0,\theta_0;\epsilon)\,| \,. \tag{2.70}$$

Theorem 2.3 suggests that solutions of the modified system (2.36)-(2.37) have two "time-decoupled" parts, one being rapid convergence to the slow manifold M_ϵ and the other being evolution near M_ϵ. Motivated by this observation we rewrite (2.37) in the form of (2.57) with a rapidly exponentially decaying perturbation.

$$\dot{\theta} = \epsilon F(t,p(\theta),h(t,\theta;\epsilon)) + \epsilon[\,F(t,p(\theta),z) - F(t,p(\theta),h(t,\theta;\epsilon))\,]. \tag{2.71}$$

and use it to show that the stability properties of a solution $\theta(t)$ of (2.57) are inherited by the solution $\theta(t)$, $z(t) = h(t,\theta(t);\epsilon)$ of (2.36)-(2.37). We show this for the case of a uniformly stable solution θ^* of (3.25). The first step is to recall a converse Lyapunov theorem; see, for example, Yoshizawa (1966).

Lemma 2.4: Suppose that $\theta^*(t)$ is a uniformly stable solution of (2.57), that $B(K_1,\theta^*(t)) \subseteq \Theta$ for all $t \in R$, and that $f(t,\theta,x)$ is a continuous function of t. Then, there exist $K_2 \in (0,K_1)$, a Lyapunov function $L(t,\theta)$, two strictly increasing positive functions γ_1 and γ_2, and a constant l, such that, for $t \geq t_0$, $|\theta - \theta^*| \leq K_2$, and $|\,\hat{\theta}-\theta^*\,| \leq K_2$,

$$\gamma_1(\,|\theta - \theta^*|\,) \leq L(t,\theta - \theta^*) \leq \gamma_2(\,|\theta - \theta^*|\,)\,, \quad \gamma_1(0) = \gamma_2(0) = 0 \tag{2.72}$$

$$|L(t,\theta-\theta^*) - L(t,\hat{\theta}-\theta^*)| \leq l\,|\theta-\hat{\theta}| \tag{2.73}$$

$$L'_{(2.57)}(t,\theta-\theta^*) \leq 0\,, \quad \forall\, \theta,\hat{\theta} \in \Theta\,. \tag{2.74}$$

□

From (2.71) it is clear that for (2.36)-(2.37) with $|z| \leq D_1$ and $|\theta-\theta^*| \leq K_2$, L satisfies

$$\begin{aligned} L'_{(2.36)-(2.37)}(t,\theta-\theta^*) &\leq L'_{(2.57)}(t,\theta-\theta^*) + \epsilon l\, \rho_z(D_1)\,|z-h(t,\theta;\epsilon)| \\ &\leq \epsilon l\;\; \rho_z(D_1) V(t,z-h(t,\theta);\epsilon). \end{aligned} \tag{2.75}$$

It follows from (2.59),(2.67), and (2.75) that the composite Lyapunov function

$$W(t,\theta,z) = L(t,\theta-\theta^*) + \beta\, V(t,z-h(t,\theta)) \tag{2.76}$$

with

$$\beta = \epsilon l\, \rho_z(D_1)/\alpha \quad . \tag{2.77}$$

satisfies $W'_{(2.36)-(2.37)}(t,\theta,z) \leq 0$ if $|z| \leq D_1$ and $|\theta-\theta^*| \leq K_2$. This proves the following theorem.

Theorem 2.4: Suppose that the conclusions of Lemma 2.4 and Theorem 2.3 hold and that $\epsilon \in (0,\epsilon_1]$. Choose $K_3 < 0$ such that $\gamma_1(|\theta|) \leq K_3$ implies $|\theta| \leq K_2$. Then, the solution of (2.36)-(2.37) with initial data $z(t_0) = z_0$, $\theta(t_0) = \theta_0$ for any (t_0,θ_0,z_0) in the set

$$S_0 = \{t,\theta,z : K\,|z| < D_1\,,\; W(t,\theta,z) < K_3\}\,, \tag{2.78}$$

remains in the set $S_1 = \{t,\theta,z : |z| < D_1\,,\; W(t,\theta,z) < K_3\}$ for all $t \geq t_0$ and along each solution $W'_{(2.36)-(2.37)}(t,\theta,z) \leq 0$. Moreover, for every constant $\delta > 0$, there exists $t_\delta > 0$ such that $\{t,\theta(t),z(t)\} \in S_\delta$ for all $t \geq t_0 + t_\delta$ where

$$S_\delta = \{t,\theta,z :\; z-h(t,\theta;\epsilon)| \leq \delta\,,\;\; L(t,\theta-\theta^*(t)) \leq W(t_0,\theta_0,z_0)\}\,. \tag{2.79}$$

□

This theorem shows that the uniform stability of $\theta^*(t)$ as a solution of (2.57) combined with the existence of an exponentially attractive integral manifold M_ϵ implies that $z^*(t) = h(t,\theta^*(t);\epsilon)$, $\theta^*(t)$ is a uniformly stable solution of (2.36)-(2.37). Since $(t,\theta,z) \in S_1$ implies that $\theta \in \Theta$, Theorem 2.4 can be translated to the original system (2.1)-(2.2).

Corollary 2.4 : Suppose that the hypotheses of Theorem 2.4 hold. Then, the solution of (2.1)-(2.2) with initial data $x(t_0) = x_0$, $\theta(t_0) = \theta_0$ for any t_0, θ_0, x_0 in the set

$$S_0 = \{t,\theta,x : K\,|\,x - \nu(t,\theta)\,| \;<\; D_1\,,\;\; W(t,\theta,x-\nu(t,\theta)) < K_3\} \tag{2.80}$$

remains in the set $S_1 = \{t,\theta,x : |\,x-\nu(t,\theta)\,| < D_1\,,\; W(t,\theta,x-\nu(t,\theta)) < K_3\}$ for all $t \leqslant t_0$ and along each such solution $W'_{(2.1)-(2.2)}(t,\theta,x-\nu(t,\theta)) \leqslant 0$. Moreover, for every constant $\delta > 0$, there exists $t_\delta > 0$ such that

$$S_\delta = \{t,\theta,x : |\,x-g(t,\theta;\epsilon)\,| \leqslant \delta\,,\;\; L(t,\theta-\theta^*(t)) \leqslant W(t_0,\theta_0,x_0)-\nu(t_0,\theta_0)\}\,. \tag{2.81}$$

□

Remark 2.9: Whereas results similar to those of Theorem 2.4 are usually obtained under the more restrictive assumption of the uniform asymptotic stability of $\theta^*(t)$ as a solution of (2.57), in Theorem 2.4 $\theta^*(t)$ is only assumed to be uniformly stable. This stronger result which is due to the fact that θ^* is a solution of the exact manifold equation (2.57), rather than an approximation, is of conceptual interest. The hypothesis of Lemma 2.4 may not be verifiable which limits the direct applicability of the result.

□

Remark 2.10: The same proof technique can be applied to show that uniform asymptotic stability or exponential stability of $\theta^*(t)$ implies uniform asymptotic stability or exponential stability, respectively, of $x^*(t) = g(t,\theta^*(t);\epsilon)$, $\theta^*(t)$. Similar techniques apply when $\theta^*(t)$ is replaced by an invariant set. The use of a quadratic Lyapunov function in place of W may provide better estimates of the region of attraction; see Saberi and Khalil (1981).

□

Remark 2.11: It may not be necessary to find the manifold function $h(t,\theta;\epsilon)$ in order to determine a solution $\theta^*(t)$ of (2.57). For example, if (2.36)-(2.37) has an equilibrium at $z = 0$, $\theta = \theta^e$, then $\theta^*(t) \equiv \theta^e$ is a solution of (2.57).

□

In specific case studies a more elaborate construction can lead to an estimate of the domain of attraction less conservative than S_0 in Theorem 2.4.

Example 2.3: Consider again Example 2.1 with constant $r=1$, $y_m = 0.5$. Let Θ_i be a segment

$[\lambda_i, \Lambda_i] \subseteq \Theta$. Similarly to Lemma 2.3 we can show that if $|z| \leqslant D_1$ and $\theta \in \Theta_i$, then the derivative of $V(z - h(\theta);\epsilon) = |z - h(\theta;\epsilon)|$ satisfies

$$V'_{(2.36)-(2.37)}(z-h(\theta;\epsilon)) \leqslant -\left[\lambda_i - 1 - \epsilon\left(\frac{1-\mu}{(\lambda_i-1)^2} + \Delta\right)\left(\frac{3-\lambda_i}{2(\lambda_i-1)} + \frac{2D+D_1}{1-\mu\Lambda_i}\right)\right]\frac{V(z-h(\theta;\epsilon))}{1-\mu\theta} \tag{2.82}$$

and the derivative of $L(\theta) = |\theta - 3|$ satisfies

$$L'_{(2.36)-(2.37)}(\theta) \leqslant L'_{(2.57)}(\theta) + \epsilon\left(\frac{3-\lambda_i}{2(\lambda_i-1)} + \frac{2D+D_1}{1-\mu\Lambda_i}\right)\frac{V(z-h(\theta;\epsilon))}{1-\mu\theta}. \tag{2.83}$$

For small enough ϵ the right-hand side of (2.82) is negative. We use $\mu = .25$, $\epsilon = .1$. Taking W as an appropriate linear combination of V and L, we can achieve $W'_{(2.36)-(2.37)}(\theta,z) \leqslant L'_{(2.57)}(\theta) < 0$ if $L(\theta) \geqslant .09$. Choosing different linear combinations of V and L for different subsets of Θ_i of Θ we create a comparison function

$$W(\theta,z) = \frac{1}{3.66}\left(m_i L(\theta) + V(z - h(\theta;\epsilon)) + \gamma_i\right) \quad \text{if} \theta \in \Theta_i \tag{2.84}$$

with m_i, γ_i, and Θ_i listed in Table 2.1. The constants γ_i are chosen so that $W(\theta,z) = c_1$, a constant, is the boundary of a compact set. This construction is such that if $c_1 < c_2$, then $\{\theta,z : W(\theta,z) \leqslant c_1\} \subset \{\theta,z : W(\theta,z) < c_2\}$. The m_i's are chosen so that $W'_{(2.36)-(2.37)}(\theta,z) \leqslant L'_{(2.57)}(\theta) < 0$ if $L(\theta) \geqslant .09$ and $m_i = 0$ if $L(\theta) \leqslant .09$. Thus any solution of (2.36)-(2.37) with $(\theta_0,z_0) \in S_0 = \{\theta,z : W(\theta,z) \leqslant 0.75\}$ remains in S_0 for all $t \geqslant t_0$ and converges in

Table 2.1. Parameters of $W(\theta,z)$

$\lvert i \rvert$	Θ_{-i}	Θ_i	$m_i = m_{-i}$	$\gamma_i = \gamma_{-i}$
1	[2.91,3.00]	[3.00,3.09]	0.00	1.143
2	[2.85,2.91]	[3.09,3.15]	1.80	0.981
3	[2.75,2.85]	[3.15,3.25]	1.92	0.963
4	[2.65,2.75]	[3.25,3.35]	2.04	0.933
5	[2.55,2.65]	[3.35,3.45]	2.20	0.877
6	[2.45,2.55]	[3.45,3.55]	2.39	0.7914
7	[2.35,2.45]	[3.55,3.65]	2.73	0.6045
8	[2.25,2.35]	[3.65,3.75]	3.66	0.00

finite time to $S_\delta = \{\theta, z : |\theta - 3| \leqslant 0.09, \ |z - h(\theta;\epsilon)| \leqslant \delta\}$. By construction $(\theta, z) \in S_0 \Rightarrow \theta \in \Theta$; hence, these solutions can be related to solutions of (2.26)-(2.27) with $x = \nu(\theta) + z$ as in Corollary 2.4. The trajectories in Fig. 2.3 which begin from the vertices of the polygon $W(\theta, x - \nu(\theta)) = 0.75$ clearly show that the equilibrium $\theta = 3$, $x = \nu(3) = 0.375$ has a domain of attraction containing S_0. For $\theta \leqslant 3$ the trajectories cross the boundary of S_0 almost perpendicularly indicating that in this region S_0 is a conservative estimate of the domain of attraction. However, this estimate is designed to guarantee that $\theta(t) \leqslant 3.75$ for all $t \geqslant t_0$, and is much less conservative for $\theta > 3$. This can be seen from the trajectory (a) converging to the manifold and the close-by divergent trajectory (b).

2.5. Attractive Integral Manifolds of a Model Reference Adaptive Control System

In this section we put a model reference adaptive control system (MRAS) in the form of (2.1)-(2.2). Then, in order to show that the MRAS possesses a slow manifold, it is sufficient to show that Assumptions 2.1, 2.2, and 2.3 are not restrictive.

As the first two assumptions are concerned only with properties of (2.1) for constant values of θ, we postpone the specification of the parameter update law (2.2) until later. The controller parametrization of Narendra and Valavani (1978) is common to several MRASs; a block diagram of the controlled system is shown in Fig. 2.4. Assuming, for ease of exposition, that the plant is strictly proper, the controlled system is described by (2.1) with

$$\begin{gathered}
A(\theta) = A_0 + b^1\theta^T \begin{bmatrix} 0 \\ C \end{bmatrix}, \ B(\theta) = [\theta_0 b^1 \mid b^3 \mid \theta_1 b^1 + b^2] \\
A_0 = \begin{bmatrix} A_p \\ 0 \\ b_f h_p \end{bmatrix}, \ b^1 = \begin{bmatrix} b_p \\ b_f \\ 0 \end{bmatrix}, \ b^2 = \begin{bmatrix} 0 \\ 0 \\ b_f \end{bmatrix}, \ b^3 = \begin{bmatrix} b_p \\ 0 \\ 0 \end{bmatrix} \\
C = \begin{bmatrix} h_p^T \\ 0 \\ 0 \end{bmatrix}, \ x = \begin{bmatrix} x_p \\ v^1 \\ v^2 \end{bmatrix}, \ w(t) = \begin{bmatrix} r(t) \\ n_1(t) \\ n_2(t) \end{bmatrix}, \ \theta = \begin{bmatrix} c_0 \\ d_0 \\ c \\ d \end{bmatrix}
\end{gathered} \tag{2.85}$$

where r is the reference input, and n_1, n_2 are disturbances and without loss of generality $h_p' = 1$. We get the Narendra and Valavani controller designed for a plant of relative degree one and order n by assigning the dimensions $c_0 \in R$, $d_0 \in R$, $c \in R^{n-1}$, $d \in R^{n-1}$, $v^1 \in R^{n-1}$, $v^2 \in R^{n-1}$

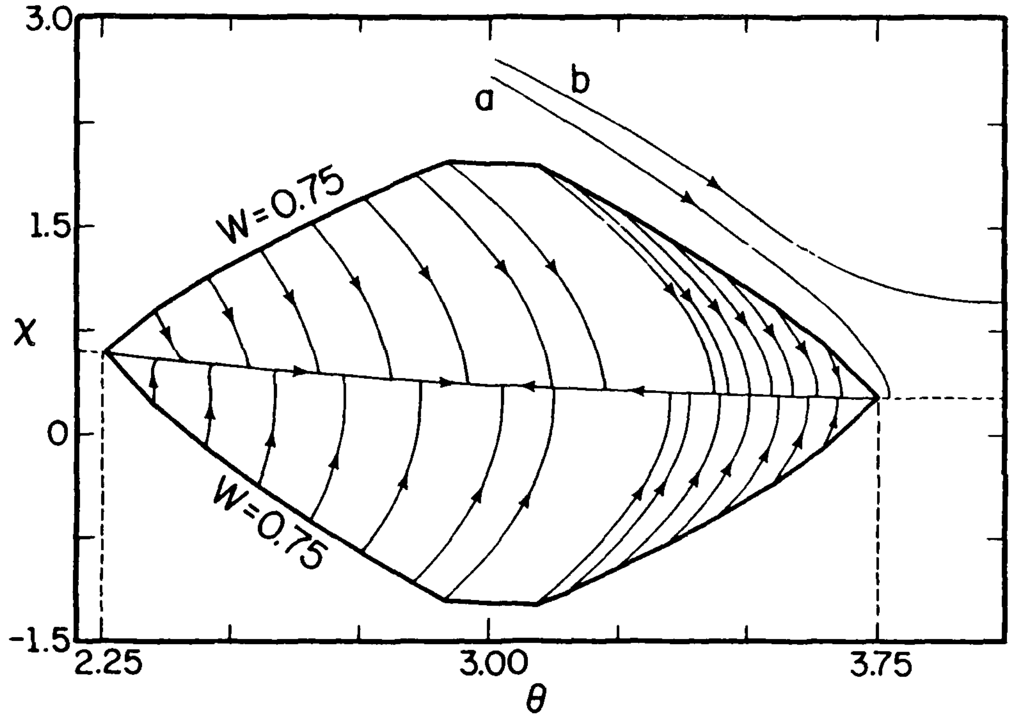

Fig. 2.3. The trajectories of (2.26)-2.27) beginning at the vertices of $W(\theta, x-\nu(\theta)) = 0.75$ converge to the equilibrium. Along trajectory (a) θ leaves the set $\Theta = [\,2.25 \;\; 3.75\,]$, but converges to the manifold, and then, to the equilibrium. Initially nearby trajectory (b) is divergent.

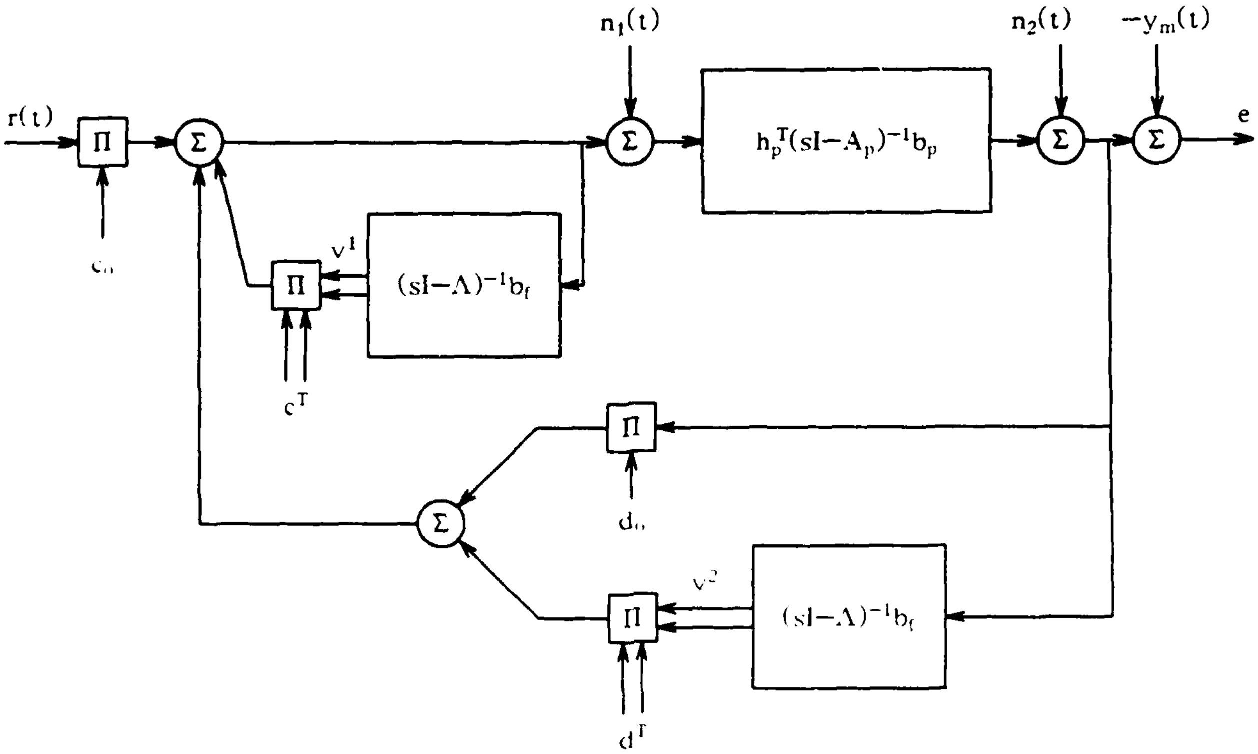

Fig. 2.4. Block diagram of the Narendra and Valavani (1978) controller parametrization.

and choosing the eigenvalues of Λ to be the zeros of the reference model. In order to show that an integral manifold exists we require A, B, and w to satisfy Assumptions 2.1 and 2.2.

Assumption 2.1 implies the stabilizability of the unknown plant by the chosen controller. In order to see that it is not very restrictive we state a proposition.

Proposition 2.1: If there exist $\theta^0 \in R^{n_\theta}$ and $\alpha_0' > 0$ such that

$$\mathrm{Re}\,\lambda(A(\theta^0)) < -\alpha_0' \tag{2.86}$$

then Assumption 2.1 is satisfied.

Proof From (2.86) it follows that $\exp[tA(\theta^0)] \leqslant K \exp[-\alpha_0' t]$ for some $K < \infty$. From standard arguments about the stability of perturbed linear systems, it follows that Assumption 2.1 is satisfied with $\alpha_0 = \alpha_0'/2$ and $\Theta = \{\theta \in R^M : |\theta-\theta^0| \leqslant K|b^1|/(2\alpha_0')\}$.

□

Thus our stabilizability assumption does not require the knowledge of the plant order, nor does it require the plant to match a reference model. We shall take the hypothesis of Proposition 2.1 as a hypothesis of Theorem 2.5. However, α_0, K, and Θ are important quantities in the analysis of the previous sections and estimating them via the proof of Proposition 2.1 is, in general, very conservative. We suggest that they be estimated in an off-line analysis via analytic expressions, simulation, or experimentation.

The following proposition shows that Assumption 2.2 is not restrictive.

Proposition 2.2: If Assumption 2.1 holds, and w(t) is uniformly bounded, piecewise Lipschitz continuous, and there exists $\delta > 0$ such that all points of discontinuity are separated by at least δ, then Assumption 2.2 is satisfied.

Proof: The boundedness and regularity of w(t) combined with Assumption 2.1 are sufficient to guarantee that $\nu(t,\theta)$ is a Lipschitz continuous function of time. Letting $\|r\| = \sup_t |r(t)|$ we can

compute bounds on v, v_1, and v_2 as

$$\begin{aligned} v &\leq \frac{K}{\alpha_0} \sup_{\theta \in \Theta} \| \theta_0 b^1 r + b^3 n_1 + (\theta_1 b^1 + b^2) n_2 \| \\ v_1 &\leq \frac{K}{\alpha_0} \, | b^1 | \, [\, \| r \| + v + \| n_2 \| \,] \qquad (2.87) \\ v_2 &\leq 2 \frac{K}{\alpha_0} \, | b^1 | \, v_1 . \end{aligned}$$

□

Thus Assumption 2.2 is a regularity and boundedness assumption on the external inputs to the system. An important observation concerning the applicability of this analysis in the design of adaptive control schemes is that these assumptions are stated for the system with constant parameters. These are assumptions about the chosen controller structure and the signals expected to enter the controlled system.

While the previous two assumptions were independent of the parameter update law, the last assumption depends only upon the parameter update law. For the MRAS which we are considering, the update law is given by (2.2) with

$$f(t,\theta,x) = f(t,x) = -\left\{ \begin{bmatrix} r \\ Cx \end{bmatrix} + \begin{bmatrix} 0 \\ n_2 \\ 0 \end{bmatrix} \right\} (h_p^T x_p + n_2 - y_m(t)) \quad . \qquad (2.88)$$

where

$$y_m(t) = \int_{-\infty}^{t} h_m^T \exp[A_m(t-\tau)] b_m r(\tau) d\tau \quad . \qquad (2.89)$$

$A_m \in R^{n \times n}$ is Hurwitz, and $W_m(s) = h_m^T (sI - A_m)^{-1} b_m$ is strictly positive real (SPR). The update law must satisfy Assumption 2.3. It is straightforward to establish that $|x| \leq D$ implies that

$$| f(t,x) | \leq \rho_F'(D) = (\| r \| + \| n_2 \| + D)(\| n_2 - y_m \| + D) \quad . \qquad (2.90)$$

$$\left| \frac{\partial f}{\partial x}(t,x) \right| \leq \rho_x(D) = \| r \| + \| n_2 \| + \| n_2 - y_m \| + 2D \quad . \qquad (2.91)$$

Then, under Assumption 2.2, Assumption 2.3 is satisfied with

$$\rho_F(D) \leqslant \rho_F'(v+D)\,,\ \rho_z(D) \leqslant \rho_x(v+D)\,,\ \rho_\theta(D) \leqslant v_1\rho_x(v+D)\,. \tag{2.92}$$

Hence, we have the following result concerning the existence of an attractive integral manifold M_ϵ for the MRAS described by (2.1)-(2.2), (2.85), (2.88)-(2.89).

Theorem 2.5: Suppose that the hypotheses of Propositions 2.1 and 2.2 hold. Then, with A,B, w, and f given by (2.85), (2.88)-(2.89), the conclusions of Theorem 2.3 hold for the modified system (2.34)-(2.35) and the conclusions of Corollary 2.3 hold for (2.1)-(2.2).

□

Remark 2.12: To this point we have made no assumptions about persistent excitation, sufficient richness, periodicity or almost periodicity of the signals entering the adaptive system.

□

Remark 2.13: If $w(t)$ and $f(t,\theta,x)$ are periodic (almost periodic) in t, then $\nu(t,\theta)$ and $h(t,\theta;\epsilon)$ are periodic (almost periodic) in t.

□

2.6. Stability in the Manifold: Averaging

In Sections 2.3 and 2.4 we derived conditions for the existence of an attractive local integral manifold M_ϵ of (2.1)-(2.2) and showed that the stability properties of a solution $\theta(t)$ of

$$\dot{\theta} = \epsilon f(t,\theta,g(t,\theta;\epsilon)) \tag{2.93}$$

which remains in Θ for all $t \geqslant t_0$ are also the stability properties of the solution $x(t) = g(t,\theta(t);\epsilon)$, $\theta(t)$ of (2.1)-(2.2). We have established these conclusions without recourse to an averaging argument. However, equation (2.93) is in the standard Bogoliubov form for the method of averaging, Hale (1980), Meerkov (1973), Sethna and Moran (1968), Volosov (1962), Bogoliubov and Mitropolski (1961). Although averaging is not the only means for analyzing (2.93), it is the one which we shall apply. The method of averaging gives very strong results for (2.93) in general, and especially, when (2.1)-(2.2) represent an adaptive system as in Section 2.5, including

(i) analysis with enough precision to provide sufficient conditions for instability as well as sufficient conditions for stability, and

(ii) interpretation in terms of the frequency spectrum of the signals entering the system and certain transfer functions in the adaptive system.

The method of averaging relates solutions of (2.93) to solutions of

$$\frac{d\overline{\theta}}{d\tau} = \overline{f}(\overline{\theta}) \tag{2.94}$$

where $\tau = \epsilon(t-t_0)$ is the slow time scale and using $g(t,\theta;0) = \nu(t,\theta)$. $\overline{f}(\theta)$ is the average of f with θ constant defined by

$$\overline{f}(\theta) = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} f(s,\theta,\nu(s,\theta))ds \,. \tag{2.95}$$

We assume w(t) and $f(t,\theta,x)$ are almost periodic in t so that the limit in (2.95) exists uniformly with respect to t. Because (2.94) is time-invariant and independent of ϵ, it is easier to study both analytically and experimentally via computer simulation, than (2.93).

Three theorems from the method of averaging lead to immediate and useful results for adaptive systems. One theorem gives conditions under which the existence of a constant solution of (2.94) implies the existence of an almost periodic solution to (2.93). This theorem also relates the stability or instability of the constant solution of (2.94) to the almost periodic solution of (2.93). It is the essential part of the theorem used to establish a stability-instability criterion for adaptive systems in Riedle and Kokotovic (1985) and Kokotovic, Riedle, and Praly (1985). The other two theorems give conditions under which the solution of (2.94) approximates the solution of (2.93). The first of these applies on a finite time interval;hence, it applies when solutions of (2.93) or (2.94) leave Θ in finite time. This result was used by Astrom (1983,1984) in his explanation of the instability mechanisms in a model reference adaptive control system. The second approximation theorem applies on infinite intervals. We shall use it to provide sufficient conditions for the uniform asymptotic stability of an almost periodic solution of (2.93) and to provide an estimate of

the region of attraction which is not restricted by linearization.

We consider first the finite interval approximation theorem

Theorem 2.6: Suppose that Assumptions 2.1, 2.2, and 2.3 hold. If the solution $\bar{\theta}(\tau)$ of (2.94) with $\bar{\theta}(0) = \theta(t_0)$ and its σ-neighborhood are in Θ for all $\tau \in [0,\tau_1)$, that is, if

$$B(\sigma,\bar{\theta}(\tau)) \subseteq \Theta \quad \forall \tau \in [0,\tau_1) \tag{2.96}$$

for any $\tau_1 \in (0,\infty)$ and any $\sigma > 0$, then there exists $\epsilon_*(\tau_1,\sigma) \in (0,\epsilon_1]$ such that for each $\epsilon \in (0,\epsilon_*)$

$$|\theta(t) - \bar{\theta}(\epsilon(t-t_0))| < \sigma \quad \forall t \in [t_0,t_0 + \tau_1/\epsilon). \tag{2.97}$$

□

Let $\Theta_1 \subset \Theta$ be a set and $\sigma_1 > 0$ a constant such that $\theta \in \Theta_1$ implies $B(\sigma_1,\theta) \subseteq \Theta$.

Corollary 2.5: Suppose that Assumptions 2.1, 2.2, and 2.3 hold and that every solution of (2.94) with $\bar{\theta}(0) \in \Theta_2 \subseteq \Theta_1$ leaves Θ_1 before τ_1. Then there exists $\epsilon_* \in (0,\epsilon_1]$ such that for each $\epsilon \in (0,\epsilon_*)$ and for any $t_0 \in R$, every solution of (2.93) with $\theta(t_0) \in \Theta_2$ leaves Θ_1 before $t_0 + \tau_1/\epsilon$.

□

Remark 2.14: The phrase "solution of (2.93) with $\theta(t_0) \in \Theta_2$" can of course be replaced by "solution of (2.1)-(2.2) with $\theta(t_0) \in \Theta_2$ and $x(t_0) = g(t_0,\theta(t_0);\epsilon)$." Using the exponential decay of $|x - g|$ one can modify the proof of Theorem 2.6 to show that off-manifold initial values of x are also allowed.

□

If the averaged system (2.94) has an instability mechanism which causes its solutions to escape in finite time from Θ_1, then this result shows that an explanation of the instability of (2.94) is a valid explanation of the predicted instability of the original system (2.93) for sufficiently slow adaptation.

Rather than repeat more of the general averaging theorems which apply to (2.93) and can be found in the cited literature, we illustrate the use of averaging theorems in the analysis of the particular MRAS presented in Section 2.5. We first use the theorem on existence and stability of almost periodic solutions of (2.93) to strengthen, by a precise definition of the tuned parameter, our

stability criterion. The weakness of the criterion was the arbitrariness of the choice of the tuned value of the parameter θ around which the system (2.1)-(2.2) was linearized in Riedle and Kokotovic (1985) and Kokotovic, Riedle, and Praly (1985). In terms of the update law (2.14), the function f in (2.93) is

$$f(t,\nu(t,\theta)) = -\phi(t,\theta)e(t,\theta) \tag{2.98}$$

where $\phi(t,\theta)$ and $e(t,\theta)$ are, respectively, the values of the regressor vector and the tracking error for constant values of θ:

$$\phi(t,\theta) = \begin{bmatrix} r(t) \\ C\nu(t,\theta) \end{bmatrix}, \quad e(t,\theta) = [h_p^T\ 0\ 0]\nu(t,\theta) + n_2(t) - y_m(t)\,. \tag{2.99}$$

Using the regressor vector $\phi(t,\theta)$ as the input to the transfer function

$$W_{CL}(\overline{\theta},s) = [h_p^T\ 0\ 0](sI - A(\overline{\theta}))^{-1}b^1\,, \tag{2.100}$$

results in a vector

$$v(t,\theta,\overline{\theta}) = W_{CL}(\overline{\theta},s)\phi(t,\theta)\,. \tag{2.101}$$

This vector is important because when $\overline{\theta} = \theta$, it can be shown that v is the sensitivity of $e(t,\theta)$ with respect to θ, that is,

$$e_\theta(t,\theta) \triangleq \frac{\partial e}{\partial \theta}(t,\theta) = v^T(t,\theta,\theta)\,. \tag{2.102}$$

By the mixed notation t,s in (2.101), we mean that the ith component of v is the almost periodic response of the closed-loop transfer function $W_{CL}(\overline{\theta},s)$ to the almost periodic ith component of ϕ. Letting θ^o be an arbitrary "tuned" parameter, we derive the equation from which the stability criterion was developed. Linearizing (2.94) around $\overline{\theta} = \theta^o$ we get

$$\frac{d\overline{\theta}}{d\tau} = -\{avg[\phi(\cdot,\theta^o)v^T(\cdot,\theta^o,\theta^o)] + avg[\phi_\theta(\cdot,\theta^o)e(\cdot,\theta^o)]\}(\overline{\theta}-\theta^o) - avg[\phi(\cdot,\theta^o)e(\cdot,\theta^o)] \tag{2.103}$$

where

$$\operatorname{avg}[e(\cdot,\theta^{o})] \triangleq \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T} e(t,\theta^{o})dt .$$

Our previous analysis, (Riedle and Kokotovic ,1985, and Kokotovic, Riedle, and Praly ,1985), neglected the terms containing $e(t,\theta^{o})$ and investigated the linear system

$$\frac{d\overline{\theta}}{d\tau} = -\operatorname{avg}[\phi(\cdot,\theta^{o})v^{T}(\cdot,\theta^{o},\theta^{o})](\overline{\theta}-\theta^{o}) . \tag{2.104}$$

Its equilibrium $\overline{\theta} = \theta^{o}$ is exponentially stable or unstable depending on the eigenvalues of $R(\theta^{o},\theta^{o})$ where

$$R(\theta,\overline{\theta}) \triangleq \operatorname{avg}[\phi(\cdot,\theta)v^{T}(\cdot,\theta,\overline{\theta})] . \tag{2.105}$$

The stability criterion on (2.104) was practical because the eigenvalue condition on the matrix $R(\theta^{o},\theta^{o})$ is easily interpretable in terms of signals and transfer functions in the system (2.1)-(2.2). Consider, for example, the case when $\phi(t,\theta)$ is the sum of a finite number of sinusoids

$$\phi(t,\theta) = \sum_{\omega\in\Omega} \psi(\theta,\omega)\, e^{j\omega t} . \tag{2.106}$$

Because ϕ is real valued, $\omega\in\Omega$ implies $-\omega\in\Omega$ and $\psi(\theta,-\omega)$ is the complex conjugate of $\psi(\theta,\omega)$. Then we compute $v(t,\theta,\overline{\theta})$ and $R(\theta,\overline{\theta})$

$$v(t,\theta,\overline{\theta}) = \sum_{\omega\in\Omega} \psi(\theta,\omega)\, W_{CL}(\overline{\theta},j\omega)\, e^{j\omega t} \tag{2.107}$$

$$R(\theta,\overline{\theta}) = \sum_{\omega\in\Omega} \psi(\theta,-\omega)\, \psi^{T}(\theta,\omega)\, W_{CL}(\overline{\theta},j\omega) . \tag{2.108}$$

Hence, the matrix $R(\theta^{o},\theta^{o})$ is easily computed if we know the Fourier series representation of $\phi(t,\theta^{o})$ and the transfer function $W_{CL}(\theta^{o},j\omega)$. An interesting sufficient condition for (2.104) to be exponentially stable is that $R(\theta^{o},\theta^{o}) + R^{T}(\theta^{o},\theta^{o})$ be positive definite, that is,

$$0 < R(\theta^{o},\theta^{o}) + R^{T}(\theta^{o},\theta^{o}) = \sum_{\omega\in\Omega} \psi(\theta^{o},-\omega)\, \psi^{T}(\theta^{o},\omega)\, \operatorname{Re} W_{CL}(\theta^{o},j\omega) . \tag{2.109}$$

The condition (2.109) has been called "signal-dependent SPR" (Riedle and Kokotovic, 1985, and Kokotovic, Riedle, and Praly, 1985) because it resembles the usual strict positive realness

requirement that $\mathrm{Re}\, W_{CL}(\theta^o, j\omega) > 0$ for all ω, but relaxes this requirement by incorporating information about the signals entering the adaptive system.

In this analysis we do not neglect $e(t,\theta^o)$ but instead assume that an equilibrium $\overline{\theta}^o$ of the averaged equation (2.94) exists, that is,

$$0 = \overline{f}(\overline{\theta}^o) = \mathrm{avg}[\phi(\cdot,\overline{\theta}^o)e(\cdot,\overline{\theta}^o)]\,. \tag{2.110}$$

Then we select $\theta = \overline{\theta}^o$ to be the tuned parameter θ^o. With this choice, the forcing term in the linearized equation (2.103) is zero and (2.103) is the linearization of the averaged system (2.94) around its equilibrium. Clearly, the stability or instability of this equilibrium is determined by the eigenvalues of

$$\overline{f}_\theta(\theta^o) = \frac{\partial \overline{f}(\theta^o)}{\partial \theta} = -R(\theta^o,\theta^o) - R_1(\theta^o) \tag{2.111}$$

$$R_1(\theta^o) = \mathrm{avg}[\phi_\theta(\cdot,\theta^o)e(\cdot,\theta^o)]\,. \tag{2.112}$$

In addition to the easily interpretable matrix R, a stability criterion for (2.103) must deal with R_1, which is much more difficult to interpret. Since a criterion for stability and instability based on the eigenvalues of $R + R_1$ is much less appealing than the criterion for (2.104), our goal is to formulate sufficient conditions for stability and instability in terms of the RMS error

$$E(\theta) = \{\mathrm{avg}[e^2(\cdot,\theta)]\}^{1/2}\,. \tag{2.113}$$

which will appear in a bound for R_1, and the eigenvalues of $R(\theta^o,\theta^o)$.

In the noncritical case, that is, when no eigenvalues of $R(\theta^o,\theta^o)$ have zero real parts, we can always find a transformation $T(\theta^o)$ such that

$$T(\theta^o)\, R_1(\theta^o,\theta^o)\, T^{-1}(\theta^o) = \begin{vmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{vmatrix}, \quad |T^{-1}(\theta^o)| \leqslant 1\,, \tag{2.114}$$

where all the eigenvalues of Λ_+ have positive real parts and all the eigenvalues of Λ_- have negative real parts. Furthermore, there exist positive constants m and λ such that

$$|\exp\left\{t\begin{bmatrix}-\Lambda_+ & 0\\ 0 & \Lambda_-\end{bmatrix}\right\}| \leqslant me^{-\lambda t}. \tag{2.115}$$

Theorem 2.7: Suppose that Assumptions 2.1, 2.2, and 2.3 hold. Let $\theta^o = \overline{\theta}^o$ be an equilibrium of the averaged system (2.94) in the interior of Θ. Suppose that no eigenvalues of $R(\theta^o,\theta^o)$ have zero real parts. Let $T(\theta^o)$ be a transformation satisfying (2.114) and m,λ be constants such that (2.115) holds. If the RMS error (2.113) is bounded by

$$E(\theta^o) < \frac{\lambda\,|T(\theta^o)|}{m\,|||\phi_\theta(\cdot,\theta^o)|||}, \tag{2.116}$$

where $\phi_\theta = \dfrac{\partial\phi}{\partial\theta}$ and

$$|||\phi_\theta(\cdot,\theta^o)||| \triangleq \{\text{avg}[\,|\phi_\theta(\cdot,\theta^o)|^2]\}^{1/2}, \tag{2.117}$$

then there exists $\epsilon_* \in (0,\epsilon_1]$ such that for each $\epsilon \in (0,\epsilon_*]$, the original system (2.93) possesses a unique almost periodic solution $\theta^*(t,\epsilon)$ which tends to θ^o as ϵ tends to zero, that is, $\lim_{\epsilon\to 0} |\theta^*(t,\epsilon)-\theta^o| = 0$. Furthermore, $\theta^*(t,\epsilon)$ is uniformly asymptotically stable if all the eigenvalues of $R(\theta^o,\theta^o)$ have positive real parts (that is, the dimension of Λ_- is zero), and $\theta^*(t,\epsilon)$ is unstable if one eigenvalue of $R(\theta^o,\theta^o)$ has a negative real part (that is, the dimension of Λ_- is greater than zero).

Proof: It is sufficient to show that no eigenvalues of $R(\theta^o,\theta^o) + R_1(\theta^o)$have zero real parts and that the dimension of Λ_- is equal to the number of eigenvalues of $R(\theta^o,\theta^o) + R_1(\theta^o)$ with negative real parts. The conclusions then follow from averaging theorems such as Theorem V.3.1 of Hale (1980). Applying the transformation $\tilde{\theta} = T(\theta_o)(\overline{\theta}-\theta^o)$ to (2.103) we have

$$\frac{d}{d\tau}\tilde{\theta} = -\left(\begin{bmatrix}\Lambda_+ & 0\\ 0 & \Lambda_-\end{bmatrix} + T(\theta^o)R_1(\theta^o)T^{-1}(\theta^o)\right)\tilde{\theta}. \tag{2.118}$$

By the Cauchy-Schwartz inequality

$$|R_1(\theta^o)| \leqslant |||\phi_\theta(t,\theta^o)|||\, E(\theta^o) \leqslant v_1 E(\theta^o) \tag{2.119}$$

where the second inequality follows from

$$\phi_\theta(t,\theta^o) = \begin{bmatrix} 0 \\ C\nu_\theta(t,\theta) \end{bmatrix}. \tag{2.120}$$

$|C| = 1$, and Assumption 2.2. Then (2.115)-(2.116) and the fact that $\begin{bmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{bmatrix} + T(\theta^o)R_1(\theta^o)T^{-1}(\theta^o)$ is similar to $R(\theta^o,\theta^o) + R_1(\theta^o)$ imply that no eigenvalues of $\overline{f}_\theta(\theta^o) = -R(\theta^o,\theta^o) - R_1(\theta^o)$ have zero real parts and that the dimension of Λ_- is equal to the number of eigenvalues of $\overline{f}_\theta(\theta^o)$ with positive real parts.

□

In light of Corollary 2.2 and the results of Section 2.4, we have the following corollary.

Corollary 2.6: Under the conditions of Theorem 2.7, for each $\epsilon \in (0,\epsilon_*]$ the system (2.1)-(2.2) with definitions (2.85), (2.88)-(2.89) possesses a unique almost periodic solution $x^*(t,\epsilon) = g(t,\theta^*(t,\epsilon);\epsilon)$, $\theta^*(t,\epsilon)$ in a neighborhood of $\nu(t,\theta^o)$, θ^o. Furthermore, this solution is u.a.s. if all the eigenvalues of $R(\theta^o,\theta^o)$ have positive real parts and unstable if one eigenvalue of $R(\theta^o,\theta^o)$ has a negative real part.

□

Although Theorem 2.7 and Corollary 2.6 are also based on linearization, this result is more complete than the original stability criterion results of Riedle and Kokotovic (1985) and Kokotovic, Riedle, and Praly (1985). It is more complete because the choice of the tuned parameter $\theta^o = \overline{\theta}^o$ allows the conclusions to apply to the actual MRAS (2.1)-(2.2). However, by itself, the choice of the tuned parameter $\theta^o = \overline{\theta}^o$ does not provide sufficient guidance for the design and analysis of an MRAS. The existence of $\overline{\theta}^o$ is not obvious and, except for the case $E(\theta^o) = 0$, defining the tuned parameter θ^o as the solution of (2.110) does not give a characterization of θ^o which is easily interpreted or checked in terms of the properties of the controlled system (2.1) with constant θ.

In the ideal case, no disturbances and no unmodeled dynamics, the parameter update law drives the tracking error to zero. An assumption of persistent excitation on the regressor vector then implies that the function $E(\theta)$ has a unique global minimum with $\min E(\theta) = 0$. In the nonideal case we are studying, suppose that the function $E(\theta)$ has an isolated local minimum at $\overline{\theta}^*$, an interior point of Θ. As the existence and properties of $\overline{\theta}^*$ are more easily checked and understood than those of $\overline{\theta}^\circ$, our next goal is to derive conditions under which the existence of $\overline{\theta}^*$ implies that $\overline{\theta}^\circ$ exists and is close to $\overline{\theta}^*$. We rewrite f from the update law (2.93) as

$$f(t,\nu(t,\theta)) = -\phi(t,\theta)[v^T(t,\theta,\overline{\theta}^*)(\theta-\overline{\theta}^*) + e(t,\overline{\theta}^*)] . \tag{2.121}$$

Using (2.121) to rewrite $\overline{f}$ of the averaged system (2.94) as

$$\overline{f}(\theta) = -R(\theta,\overline{\theta}^*)(\theta-\overline{\theta}^*) - \mathrm{avg}[\phi(\cdot,\theta)e(\cdot,\overline{\theta}^*)] . \tag{2.122}$$

we state the following result.

Theorem 2.8: Suppose that $\overline{\theta}^*$, an interior point of Θ, provides an isolated local minimum $E(\overline{\theta}^*)$ of $E(\theta)$ and that $R(\overline{\theta}^*, \overline{\theta}^*)$ is nonsingular. Under these conditions there exist $\mu_0 > 0$ and $\gamma_0 < \infty$ such that if $E(\overline{\theta}^*) < \mu_0$, then the equilibrium $\overline{\theta}^\circ$ exists and

$$|\overline{\theta}^\circ - \overline{\theta}^*| \leqslant \gamma_0 E(\overline{\theta}^*) . \tag{2.123}$$

Proof: Factoring $e(t,\overline{\theta}^*)$ as the product of $e_1(t)$, with unit amplitude, and μ, a scaling factor,

$$e(t,\overline{\theta}^*) = \mu e_1(t) , \quad \mathrm{avg}[e_1^2(\cdot)] = 1 \tag{2.124}$$

we prove the continuity of the function $\overline{\theta}^\circ(\mu)$ implicitly defined by

$$0 = \overline{f}^1(\theta,\mu) \triangleq \overline{\theta}_*)(\theta-\overline{\theta}^*) - \mu\, \mathrm{avg}[\phi(\cdot,\theta)e_1(\cdot)] . \tag{2.125}$$

Since $\overline{f}^1(\overline{\theta},0) = 0$, $\overline{f}_\theta^1(\overline{\theta}^*,0) = R(\overline{\theta}^*,\overline{\theta}^*)$, and $\overline{f}_\mu^1(\overline{\theta}^*,0) = \mathrm{avg}[\phi(\cdot,\overline{\theta}^*)e_1(\cdot)]$, the claim follows from the implicit function theorem.

□

With the combination of Theorems 2.1 and 2.2 we have a result which is strong enough to be practically applicable for the design of adaptive systems. Doing off-line analysis or simulation of (2.1) for constant values of parameters, we can check for typical signals $w(t)$ entering the system, whether or not $E(\theta)$ has a minimum for some value $\overline{\theta}^*$ in the interior of the set Θ. If this minimum exists we can then check the fact that the slowly adapting system has an almost periodic solution which preserves its u.a.s. property in the presence of a nonzero tracking error. It is convenient that the restrictions Theorem 2.2 places on the tracking error are for the minimum value of the RMS error $E(\overline{\theta}^*)$. However, this result is local and does not give estimates of μ_0, γ_0 or the region of attraction of the u.a.s. solution. Theorems 2.1 and 2.2 are local because their hypotheses depend on the eigenvalues of the constant matrices $R(\overline{\theta}^\circ,\overline{\theta}^\circ)$ and $R(\overline{\theta}^*,\overline{\theta}^*)$, respectively. Our next result considers $R(\theta,\overline{\theta}^*)$ as a function of θ in Θ and makes use of the infinite time approximation theorem to address these weaknesses.

Theorem 2.9: Suppose that $\overline{\theta}^*$ provides a local minimum $E(\overline{\theta}^*)$ of $E(\theta)$ in the set $\mathbf{B}(\beta,\overline{\theta}^*) \subsetneq \Theta$ and that

$$R(\theta,\overline{\theta}^*) + R^T(\theta,\overline{\theta}^*) \geqslant 2\lambda I > 0 \quad \forall\theta \in \mathbf{B}(\beta,\overline{\theta}^*) . \tag{2.126}$$

If the minimum RMS error $E(\overline{\theta}^*)$ satisfies

$$E(\theta^*) < \lambda^2/(2w_0\overline{v}_1\overline{v}^2 + \lambda\overline{v}_1) \tag{2.127a}$$

$$E(\overline{\theta}^*) < \beta\lambda/(3\overline{v}) \tag{2.127b}$$

where

$$w_0 = \frac{K}{\alpha_0} \, | b_1 |$$

$$\overline{v} = \max_{\theta \in B(\beta,\overline{\theta}^*)} ||| \phi(\cdot,\theta) ||| \leqslant v$$

$$\overline{v}_1 = \max_{\theta \in B(\beta,\overline{\theta}^*)} ||| \phi_u(\cdot,\theta) ||| \leqslant v_1$$

with v and v_1 from Assumption 2.2, then

(i) the averaged system (2.94) has a u.a.s. equilibrium $\bar{\theta}^{\circ}$ such that

$$|\bar{\theta}^{\circ}-\bar{\theta}^{*}| \leqslant \frac{\bar{v}}{\lambda}E(\bar{\theta}^{*}) . \tag{2.128}$$

(ii) every solution of (2.94) with $\bar{\theta}(0) \in \mathbf{B}(\beta,\bar{\theta}^{*})$ satisfies

$$\bar{\theta}(\tau) \in \mathbf{B}(\beta,\theta^{*}) \;\; \forall\tau \geqslant 0 , \;\; \lim_{\tau\to\infty} |\bar{\theta}(\tau)-\bar{\theta}^{\circ}| = 0 ; \tag{2.129}$$

(iii) given $\sigma > 0$ there exists $\epsilon_{*}(\sigma) \in (0,\epsilon_1]$ such that for each $\epsilon \in (0,\epsilon_{*}]$ the original system system (2.93) possesses an almost periodic solution $\theta^{*}(t,\epsilon)$ which is u.a.s. and

$$\lim_{\epsilon\to 0} |\theta^{*}(t,\epsilon) - \bar{\theta}^{\circ}| = 0 ; \tag{2.130}$$

(iv) every solution of (2.93) with $\theta(t_0,\epsilon) = \bar{\theta}(0) \in \mathbf{B}(\beta-\sigma,\bar{\theta}^{*})$ satisfies for each $\epsilon \in (0,\epsilon_{*}]$

$$\theta(t,\epsilon) \in \mathbf{B}(\beta,\theta^{*}) , \;\; |\theta(t,\epsilon)-\bar{\theta}(\epsilon(t-t_0))| < \sigma , \;\; \forall t \geqslant t_0 . \tag{2.131}$$

and

$$\lim_{t\to\infty} |\theta(t,\epsilon) - \theta^{*}(t,\epsilon)| = 0 . \tag{2.132}$$

Proof: Because the Sethna and Moran (1968) theorem, together with (i) and (ii), implies (iii) and (iv), we need only establish (i) and (ii). Define the mapping $T : \mathbf{B}(\bar{v}\lambda^{-1}E(\bar{\theta}^{*}),\bar{\theta}^{*}) \to \mathbf{B}(\bar{v}\lambda^{-1}E(\bar{\theta}^{*}),\bar{\theta}^{*})$ by

$$T(\theta) = \bar{\theta}^{*}-R^{-1}(\theta,\bar{\theta}^{*})\mathrm{avg}[\phi(\cdot,\theta)e(\cdot,\bar{\theta}^{*})] . \tag{2.133}$$

The inequalities (2.127) are sufficient for T to be a contraction mapping on $\mathbf{B}(\bar{v}\lambda^{-1}E(\bar{\theta}^{*}),\bar{\theta}^{*})$; hence, T has a unique fixed point $\bar{\theta}^{\circ} \in \mathbf{B}(\bar{v}\lambda^{-1}E(\bar{\theta}^{*}),\bar{\theta}^{*})$. Clearly, the fixed point of T is a solution of $\bar{f}(\theta) = 0$. Thus (2.94) has an equilibrium $\bar{\theta}^{\circ}$ satisfying the bound (2.128). We establish that this equilibrium is exponentially stable with the Lyapunov function $V(\theta) = \frac{1}{2}\,\theta^{T}\theta$. Rewriting (2.94) in the form

$$\frac{d}{d\tau}\bar{\theta} = -R(\bar{\theta},\bar{\theta}^*)(\bar{\theta}-\bar{\theta}^\circ) - [b(\bar{\theta})-b(\bar{\theta}^\circ)]$$
$$+ [R(\bar{\theta},\bar{\theta}^*) - R(\bar{\theta}^\circ,\bar{\theta}^*)](\bar{\theta}^\circ-\bar{\theta}^*) . \qquad (2.134)$$

where

$$b(\theta) = \mathrm{avg}[\phi(\cdot,\theta)e(\cdot,\bar{\theta}^*)] ,$$

it follows that

$$\frac{d}{d\tau} V(\bar{\theta}-\bar{\theta}^\circ) \leqslant 2[-\lambda+(2w_0\bar{v}_1\bar{v}^2\lambda^{-1} + \bar{v}_1)E(\theta^*)]V(\bar{\theta}-\bar{\theta}^\circ) \qquad (2.135)$$

for all $\bar{\theta} \in \mathbf{B}(\beta,\bar{\theta}^*)$. The inequalities (2.127b) and (2.135) imply that $\bar{\theta}^\circ$ is exponentially stable and that its region of attraction includes $\mathbf{B}(2\beta/3,\bar{\theta}^\circ)$, that is, there exists $\lambda_1 > 0$ such that solutions of (2.94) with $\theta(\tau_0) \in \mathbf{B}(2\beta/3 , \bar{\theta}^\circ) \subseteq \mathbf{B}(\beta,\bar{\theta}^*)$ for any $\tau_0 \geqslant 0$ satisfy for all $\tau \geqslant \tau_0$

$$|\bar{\theta}(\tau)-\bar{\theta}^\circ| \leqslant |\bar{\theta}(\tau_0)-\bar{\theta}^\circ| \exp[-\lambda_1(\tau-\tau_0)] . \qquad (2.136)$$

To show that the region of attraction of $\bar{\theta}^\circ$ includes $\mathbf{B}(\beta,\bar{\theta}^*)$, that is, to establish (2.129) it is enough to show that solutions of (2.94) with $\bar{\theta}(0) \in \mathbf{B}(\beta,\bar{\theta}^*)$ enter $\mathbf{B}(2\beta/3,\bar{\theta}^\circ)$ in finite time. Using the form of $\bar{f}$ in (2.122) we compute the inequality

$$\frac{d}{d\tau} V(\bar{\theta}-\bar{\theta}^*) \leqslant -\lambda|\bar{\theta}-\bar{\theta}^*|^2 + \bar{v}E(\bar{\theta}^*)|\bar{\theta}-\bar{\theta}^*| . \qquad (2.137)$$

Choosing $\delta > 0$ such that $\delta + \bar{v}\lambda^{-1}E(\bar{\theta}^*) < \beta/3$, it is clear from (2.137) that solutions of (2.94) beginning in $B(\beta,\bar{\theta}^*)$ enter $B(\delta + \bar{v}\lambda^{-1}E(\bar{\theta}^*),\bar{\theta}^*) \subseteq B(2\beta/3,\bar{\theta}^\circ)$ in finite time.

□

The stability condition (2.126) is again the signal-dependent SPR condition. However, it is now evaluated pointwise in θ for each θ in a ball around $\bar{\theta}^*$ rather than at only the point $\theta = \theta^*$. The formula (2.108) for $R(\theta,\bar{\theta})$ is still valid; hence, the frequency domain interpretations of (2.126) are analogous to those of (2.109). Finally, we point out that as θ varies over $\mathbf{B}(\beta,\bar{\theta}^*)$ the transfer function $W_{CL}(\bar{\theta}^*,s)$ does not change. That is , the condition (2.126) is a signal-dependent SPR condition on the fixed transfer function $W_{CL}(\theta^*,s)$ for the different signals $\phi(t,\theta)$ as θ varies over the ball $\mathbf{B}(\beta,\bar{\theta}^*)$.

2.7. Concluding Remarks

Pursuing the intuitively appealing distinction between states and parameters, we have established conditions for the existence of an integral manifold — the slow manifold — and used it for an exact description of the slow adaptation process. Conditions for the exponential attractivity of the slow manifold are formulated, and the exponential attractivity is shown to imply that the stability properties of a solution of the reduced-order system on the manifold are also the stability properties of the corresponding solution of the full-order system. Based on this reduced-order exact description, we have examined the validity of earlier results obtained via the averaging of what is now shown to be a "frozen parameter" approximation of the slow manifold. A particular model reference adaptive control system is shown to possess an exponentially attractive slow manifold. The stability of this system is then analyzed via averaging of the equation describing the motion in the manifold. This analysis extends and completes earlier results based on a linearization near a "tuned" system.

CHAPTER 3

INTEGRAL MANIFOLDS OF SLOW ADAPTATION IN DISCRETE TIME

3.1. Introduction

Many discrete time adaptive control schemes are described by ordinary difference equations of the form

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))w(k) \; , \tag{3.1}$$

$$\theta(k+1) = \theta(k) + \epsilon f(k,\theta(k),x(k)) \; , \tag{3.2}$$

where, as in (2.1)-(2.2), x contains the states of the plant, controller, filters, etc., and θ is the vector of adjustable parameters. We remark that if the update law is of the Newton or least-squares type, then θ contains the columns of the Newton matrix. As in Chapter 2, we first derive conditions under which (3.1)-(3.2) possesses an integral manifold M_ϵ of the form

$$M_\epsilon = \{k,\theta,x : x = g(k,\theta;\epsilon)\} \; . \tag{3.3}$$

and then consider its attractivity. Restricted to the slow manifold M_ϵ the system (3.1)-(3.2) evolves according to $x(k) = g(k,\theta(k);\epsilon)$ and

$$\theta(k+1) = \theta(k) + \epsilon f(k,\theta(k),g(k,\theta(k);\epsilon)) \; , \tag{3.4}$$

which is in the standard form for averaging. Discrete time averaging theory relates the solutions of (3.4) to the solutions of the ordinary differential equation (ODE)

$$\frac{d}{d\tau}\overline{\theta} = \overline{f}(\overline{\theta}) \tag{3.5}$$

where

$$\overline{f}(\theta) = \lim_{N\to\infty}\frac{1}{N}\sum_{j=k}^{k+N-1} f(j,\theta,g(j,\theta;0)) \; . \tag{3.6}$$

Leaving the application of this theory to a specific adaptive system for Chapter 4, we give proofs of several discrete-time averaging theorems using deterministic assumptions. By making appropriate assumptions on the stochastic process which generates the inputs to (3.1)-(3.2), we conclude this

chapter with a lemma that relates the trajectories of some stochastic adaptive control systems in the form (3.1)-(3.2) to solutions of the ODE (3.5).

3.2. Approximation of the Slow Manifold

As in the continuous-time case, an integral manifold M_ϵ of (3.1)-(3.2) is defined by the statement that if x,θ is in M_ϵ at $k=k_0$, then it is in M_ϵ for all $k \geq k_0$, that is,

$$(k_0,\theta(k_0),x(k_0)) \in M_\epsilon \Rightarrow (k,\theta(k),x(k)) \in M_\epsilon \quad \forall k \quad \forall k_0 \in \mathbf{Z} \ . \tag{3.7}$$

In general, solving for the function $g(k,\theta;\epsilon)$, which determines M_ϵ via (3.3), is as difficult as solving the complete system (3.1)-(3.2). Our approach, then, is to find an easily computable and meaningful approximation of $g(k,\theta;\epsilon)$. Note that at $\epsilon=0$, θ is constant and (3.1) is a linear time-invariant system with input $w(k)$. Hence, we can compute the solution of (3.1)-(3.2) for $\epsilon=0$. The variation of constants formula applied to (3.1) with $\theta(k) \equiv \theta$, a constant, gives

$$x(k) = A^{k-k_0}(\theta)x(k_0) + \sum_{i=k_0}^{k-1} A^{k-1-i}(\theta)B(\theta)w(i) \ , \ k \geq k_0+1 \ . \tag{3.8}$$

Assumption 3.1: There exist a set Θ and constants $K \in [1,\infty)$ and $\lambda_0 \in (0,1)$ such that

$$|A^i(\theta)| \leq K\lambda_0^i \ , \ i \geq 0 \ , \ \forall \theta \in \Theta \ . \tag{3.9}$$

□

Making use of this stability assumption, we let $k_0 \to -\infty$ in (3.8) and take the steady-state response

$$\nu(k,\theta) = \sum_{i=-\infty}^{k-1} A^{k-1-i}(\theta)B(\theta)w(i) \tag{3.10}$$

as the manifold function $g(k,\theta;0) = \nu(k,\theta)$ defining the frozen parameter manifold M_0. This function is both meaningful and easy to compute. Therefore, in addition to proving that M_ϵ exists, we adopt the goal of showing that $g(k,\theta;0) = \nu(k,\theta)$ is a good approximation of $g(k,\theta;\epsilon)$ for small ϵ. In order to meet both goals simultaneously, we introduce the deviation of x from $\nu(k,\theta)$ as a new state variable

$$z = x - \nu(k,\theta) \ . \tag{3.11}$$

and transform (3.1)-(3.2) into

$$z(k+1) = A(\theta(k))z(k) - G(k+1,\theta(k),z(k)) \tag{3.12}$$

$$\theta(k+1) = \theta(k) + \epsilon F(k,\theta(k),z(k)) \tag{3.13}$$

where

$$G(k+1,\theta,z) \triangleq \nu(k+1,\theta+\epsilon F(k,\theta,z)) - \nu(k+1,\theta) \tag{3.14}$$

$$F(k,\theta,z) \triangleq f(k,\theta,\nu(k,\theta) + z) \ . \tag{3.15}$$

The goals are met by proving that (3.12)-(3.13) possesses an integral manifold M_ϵ determined by

$$M_\epsilon = \{k,\theta,z : \ z = h(k,\theta;\epsilon)\} \tag{3.16}$$

with $h(k,\theta;\epsilon) = O(\epsilon)$.

Before proving the existence of $h(k,\theta;\epsilon)$, we consider formal (without proof) approximations of $h(k,\theta;\epsilon)$. From the definition of M_ϵ it follows that the function $h(k,\theta;\epsilon)$ evaluated along a trajectory of (3.12)-(3.13) which is in M_ϵ must satisfy (3.12)-(3.13) with z replaced by h. Performing this substitution, we get the functional difference equation

$$h(k+1,\theta+\epsilon F(k,\theta,h(k,\theta;\epsilon));\epsilon) = A(\theta)h(k,\theta;\epsilon) - G(k+1,\theta,h(k,\theta;\epsilon)) \ . \tag{3.17}$$

This is no longer an ordinary difference equation in k because of the variations in the second argument of h. Rewriting (3.17) as

$$\begin{aligned} h(k+1,\theta;\epsilon) &= A(\theta)h(k,\theta;\epsilon) - G(k+1,\theta,h(k,\theta;\epsilon)) \\ &\quad - [h(k+1,\theta+\epsilon F(k,\theta,h(k,\theta;\epsilon));\epsilon) - h(k+1,\theta;\epsilon)] \end{aligned} \tag{3.18}$$

we bring (3.17) to a discrete-time analog of the partial differential equation (2.23). Under an appropriate smoothness assumption we can approximate $h(k,\theta;\epsilon)$ by a power series in ϵ

$$h(k,\theta;\epsilon) = h_0(k,\theta) + \epsilon h_1(k,\theta) + \epsilon^2 h_2(k,\theta) + \cdots \ . \tag{3.19}$$

Substituting this series for $h(k,\theta;\epsilon)$ in (3.18) and equating the coefficients of like powers of ϵ, the equation for $h_0(k,\theta)$ is an ordinary difference equation in k whose steady-state response is zero:

$$h_0(k+1,\theta) = A(\theta)h_0(k,\theta) \Rightarrow h_0(k,\theta)\equiv 0 \quad . \tag{3.20}$$

For the ϵ term $h_1(k,\theta)$ the ordinary difference equation is

$$h_1(k+1,\theta) = A(\theta)h_1(k,\theta) - \nu_\theta(k+1,\theta)F(k,\theta,0) \quad . \tag{3.21}$$

and its steady-state solution for each fixed θ is given by

$$h_1(k,\theta) = \sum_{i=-\infty}^{k-1} A^{k-1-i}(\theta)\nu(i+1)F(i,\theta,0) \quad . \tag{3.22}$$

Here use is made of $F(k,\theta,h_0(k,\theta)) = F(k,\theta,0)$. This process of successively evaluating the h_j's continues with each coefficient $h_j(k,\theta)$ in the expansion (3.19) being the steady-state response of a linear ordinary difference equation in k, parametrically dependent on θ, and driven by terms with h_i and derivatives of h_i only for $i < j$.

3.3. Existence of a Slow Manifold

As in the continuous-time case, we derive conditions under which M_ϵ exists by constructing a map T_ϵ whose fixed point is $h(k,\theta;\epsilon)$ and finding conditions for T_ϵ to be a contraction. We first specify a closed subset of a Banach space in which to search for $h(k,\theta;\epsilon)$. Letting the space be the set of continuous functions $H(k,\theta)$ equipped with the norm $\|H\| = \sup_{k,\theta\in \mathbf{Z}\times\mathbf{R}^{n_\theta}} |H(k,\theta)|$, we use positive constants D, Δ to define our closed subset $\mathbf{H}(D,\Delta)$ as

$$\mathbf{H}(D,\Delta) = \{H:\mathbf{Z}\times R^{n_\theta}\rightarrow R^{n_x} \; \|H\|\leqslant D \text{ and } |H(k,\theta)-H(k,\hat{\theta})| \leqslant \Delta\,|\theta-\hat{\theta}|\forall k\in\mathbf{Z};\forall\theta,\hat{\theta}\in R^{n_\theta}\} \quad . \tag{3.23}$$

Recall that our goal is not only to prove that $h(k,\theta;\epsilon)$ exists but also to show that D and Δ are $O(\epsilon)$. In addition to the stability assumption we require the parameterization $A(\theta)$, $B(\theta)$ to be continuously differentiable and to have Lipschitz first derivatives for $\theta\in\Theta$ and $w(k)$ to be uniformly bounded. We quantify this requirement in the following assumption.

Assumption 3.2: There exist positive constants v, v_1, v_2 such that

$$|\nu(k,\theta)| \leqslant v, \; |\nu_\theta(k,\theta)| \leqslant v_1 \, , |\nu_\theta(k,\theta)-\nu_\theta(k,\hat{\theta})| \leqslant v_2|\theta-\hat{\theta}| \quad . \tag{3.24}$$

for all $k \in \mathbf{Z}$, and all $\theta, \hat{\theta} \in \Theta$.

□

With this assumption, the existence of $h \in \mathbf{H}(D, \Delta)$ implies that g is uniformly bounded and Lipschitz continuous in θ. Finally, we must use a parameter update law (3.2) with $F(k, \theta, x)$ being bounded and Lipschitzian in θ, x uniformly with respect to $k \in \mathbf{Z}, \theta \in \Theta$, and x in compact sets.

Assumption 3.3: There exist nondecreasing positive functions $\rho_F(D)$, $\rho_\theta(D)$, and $\rho_z(D)$ such that

$$\begin{aligned} &|F(k,\theta,z)| \leqslant \rho_F(D),\ |F(k,\theta,z)-F(k,\hat{\theta},z)| \leqslant \rho_\theta(D)|\theta-\hat{\theta}| \ , \\ &|F(k,\theta,z)-F(k,\theta,\hat{z})| \leqslant \rho_z(D)|z-\hat{z}| \end{aligned} \tag{3.25}$$

for all $k \in \mathbf{Z}$, for $\theta, \hat{\theta} \in \Theta$, and for all $z, \hat{z}$ with $|z| \leqslant D, |\hat{z}| \leqslant D$.

□

Anticipating the same type of stability conditions that were encountered in Chapter 2, we again introduce a modified system where θ is replaced by $p(\theta) \in \Theta$ in A, G, and F. Notice that $G(k+1, \theta, z)$ contains $\nu(k+1, \theta+\epsilon F(k, \theta, z))$. In order to ensure that $p(\theta)+\epsilon F(k, p(\theta), z) \in \Theta$ we must choose $p(\theta)$ strictly inside Θ. Let $\Theta_1(\epsilon, D)$ be a compact, convex subset of Θ such that $\theta \in \Theta_1(\epsilon, D)$ implies $\mathbf{B}(\epsilon\rho_F(D), \theta) \subseteq \Theta$. We take $p(\theta)$ to be the unique element of $\Theta_1(\epsilon, D)$ which is closest to θ, namely,

$$p(\theta) = \arg \min_{p \in \Theta_1(\epsilon, D)} |p-\theta| \ . \tag{3.26}$$

Remark 3.1: With this definition p is potentially a function of ϵ and D. However, the meaning of p and the choice of the set Θ_1 are generally clear. Hence, for notational simplicity we do not explicitly indicate the dependence of p on ϵ or D.

□

We henceforth analyze the modified system

$$z(k+1) = A(p(\theta(k)))z(k)-G(k+1, p(\theta(k)), z(k)) \tag{3.27}$$

$$\theta(k+1) = \theta(k)+\epsilon F(k, p(\theta(k)), z(k)) \ . \tag{3.28}$$

This modified system is similar to but not identical to the usual kind of parameter update laws incorporating projection, which have the form

$$\theta(k+1) = p[\theta(k)+\epsilon F(k,\theta(k),z(k))] \quad . \tag{3.29}$$

Update laws such as (3.29) ensure $\theta(k)\in\Theta$ for all k which makes the $p(\theta)$ in A and G unnecessary, whereas (3.28) allows $\theta(k)$ to leave Θ but $\theta(k+1)-\theta(k)$ is always computed using $p(\theta(k))\in\Theta$. The study of (3.29) introduces technical difficulties at the boundary of Θ. A topic of future research is how to avoid the technical difficulties associated with the usual projection algorithm (3.29) while avoiding the potential unboundedness associated with the modified update law (3.28).

With the use of $A(p(\theta))$, $B(p(\theta))$, $G(p(\theta))$, $F(k,p(\theta),z)$, the fact that $|p(\theta)-p(\hat{\theta})| \leqslant |\theta-\hat{\theta}|$ implies that the modified system satisfies Assumptions 3.1-3.3 for all $\theta\in R^{n_\theta}$. We describe the dependence of $F(k,p(\theta),H(k,\theta)$ and $G(k+1,\theta,H(k,\theta))$ on θ for $H\in\mathbf{H}(D,\Delta)$ by defining $\rho_1(D,\Delta)$ and $\rho_2(D,\Delta)$ such that

$$\begin{aligned} |F(k,p(\theta),H(k,\theta)) - F(k,p(\hat{\theta}))H(k,\hat{\theta}))| &\leqslant \rho_1(D,\Delta)|\theta-\hat{\theta}| \\ |G(k+1,p(\theta),H(k,\theta)) - G(k+1,p(\hat{\theta}),H(k,\hat{\theta}))| &\leqslant \epsilon\rho_2(D,\Delta)|\theta-\hat{\theta}| \end{aligned} \tag{3.30}$$

for all $k\in\mathbf{Z}$, for all $\theta,\hat{\theta}\in R^{n_\theta}$ and all $H\in\mathbf{H}(D,\Delta)$. It follows from Assumptions 3.2 and 3.3 that ρ_1 and ρ_2 exist and can be chosen to satisfy

$$\rho_1(D,\Delta) \leqslant \rho_\theta(D)+\Delta\rho_z(D), \qquad \rho_2(D,\Delta) \leqslant v_1\rho_1(D,\Delta)+v_2\rho_F(D) \quad . \tag{3.31}$$

Our first step in constructing the map T_ϵ is to define $\theta_j(i;k,\theta,\epsilon)$ for $i\leqslant k$ as the solution of the end-value problem

$$\theta_j(i+1) = \theta_j(i)+\epsilon F(i,p(\theta_j(i)),H_j(i,\theta_j(i))), \quad \theta_j(k) = \theta \quad , \tag{3.32}$$

where the subscript j implies dependence on H_j. We bound the dependence of $\theta_j(i;k,\theta,\epsilon)$ on θ and H_j in the following lemma.

Lemma 3.1: Suppose that Assumptions 3.1-3.3 hold. Let $\epsilon_1(D,\Delta) = 1/\rho_1(D,\Delta)$. For each $\epsilon\in[0,\epsilon_1)$, if $H_j\in\mathbf{H}(D,\Delta)$ and $H_m\in\mathbf{H}(D,\Delta)$, then

$$|\theta_j(i;k,\theta,\epsilon) - \theta_j(i;k,\hat{\theta},\epsilon)| \leqslant \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]^{k-i} |\theta-\hat{\theta}| \quad . \tag{3.33}$$

$$|\theta_j(i;k,\theta,\epsilon)-\theta_m(i;k,\theta,\epsilon)| \leqslant \frac{\rho_z(D)}{p_1(D,\Delta)}\left[\left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]^{k-i}-1\right]\|H_j-H_m\| \quad . \tag{3.34}$$

for all $i \leqslant k$.

Proof: Letting $\theta_j(i)$ and $\hat{\theta}_j(i)$ denote $\theta_j(i;k,\theta,\epsilon)$ and $\theta_j(i;k,\hat{\theta},\epsilon)$, respectively, we have

$$\begin{aligned}|\theta_j(i+1)-\hat{\theta}_j(i+1)| &\geqslant |\theta_j(i)-\hat{\theta}(i)|-\epsilon\rho_1(D,\Delta)|\theta_j(i)-\hat{\theta}_j(i)| \\ &= [1-\epsilon\rho_1(D,\Delta)]|\theta_j(i)-\hat{\theta}_j(i)|\end{aligned} \tag{3.35}$$

for all $i \leqslant k-1$. Dividing both sides of (3.35) by $1-\epsilon\rho_1(D,\Delta)>0$, we get the desired result.

$$\begin{aligned}|\theta_j(i)-\hat{\theta}_j(i)| &\leqslant \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]|\theta_j(i+1)-\hat{\theta}_j(i+1)| \\ &\leqslant \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]^{k-i}|\theta_j(k)-\hat{\theta}_j(k)| \\ &= \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]^{k-i}|\theta-\hat{\theta}| \quad .\end{aligned} \tag{3.36}$$

Letting $\theta_j(i)$ and $\theta_m(i)$ denote $\theta_j(i;k,\theta,\epsilon)$ and $\theta_m(i;k,\theta,\epsilon)$, respectively, we have

$$\begin{aligned}|\theta_j(i+1)-\theta_m(i+1)| \geqslant |\theta_j(i)-\theta_m(i)| &- \epsilon\rho_1(D,\Delta)|\theta_j(i)-\theta_m(i)| \\ &- \epsilon\rho_z(D)|H_j(i,\theta_m(i))-H_m(i,\theta_m(i))|\end{aligned} \tag{3.37}$$

for all $i \leqslant k-1$. We again divide by $1-\epsilon\rho_1(D,\Delta)$ to obtain the desired result.

$$\begin{aligned}|\theta_j(i)-\theta_m(i)| &\leqslant \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]|\theta_j(i+1)-\theta_m(i+1)| + \frac{\epsilon\rho_z(D)}{1-\epsilon\rho_1(D,\Delta)}\|H_j-H_m\| \\ &\leqslant \frac{\epsilon\rho_z(D)}{1-\epsilon\rho_1(D,\Delta)}\|H_j-H_m\| \sum_{n=0}^{k-i-1}\left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]^n \\ &= \frac{\epsilon\rho_z(D)}{1-\epsilon\rho_1(D,\Delta)}\|H_j-H_m\|\frac{1-\epsilon\rho_1(D,\Delta)}{\epsilon\rho_1(D,\Delta)}\left[\left[\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right]^{k-i}-1\right] \quad .\end{aligned} \tag{3.38}$$

□

The second step in the construction of the map T_ϵ requires the stability of the linear time-varying system

$$z(i+1) = A(p(\theta_j(i)))z(i) \quad . \tag{3.39}$$

where $\theta_j(i) = \theta_j(i;k,\theta,\epsilon)$ is considered as a given function of time for a given $H_j \in \mathbf{H}(D,\Delta)$. The state transition matrix of (3.39) is given by

$$\Phi_j(n_1,n_2;k,\theta,\epsilon) = \begin{cases} I & n_1 = n_2 \\ A(p(\theta_j(n_1-1)))A(p(\theta_j(n_1-2)))\cdots A(p(\theta_j(n_2))) & n_1 > n_2 \end{cases} \tag{3.40}$$

and we establish its stability in the following lemma.

Lemma 3.2: Suppose that Assumptions (3.1)-(3.3) hold. Let a be the Lipschitz constant of $A(\theta)$ for $\theta \in \Theta$, that is,

$$|A(\theta)-A(\hat{\theta})| \leqslant a|\theta-\hat{\theta}| \tag{3.41}$$

for all $\theta,\hat{\theta} \in \Theta$. Denoting the largest integer less than or equal to N by $\lfloor N \rfloor$, let

$$N(\epsilon,D) = \left\lfloor \left(\frac{4\lambda_0(k-1)}{\epsilon K a \rho_F(D)} \right)^{1/2} \right\rfloor . \tag{3.42}$$

If $H_j \in \mathbf{H}(D,\Delta)$, then

$$|\Phi_j(n_1,n_2;k,\theta,\epsilon)| \leqslant K\lambda_1^{n_1-n_2}(\epsilon,D) , \qquad n_1 \geqslant n_2 , \tag{3.43}$$

where

$$\lambda_1(\epsilon,D) = \begin{cases} K^{(1/N(\epsilon,D))}[\lambda_0+\epsilon K a \rho_F(D) N(\epsilon,D)/4] & , \text{ if } \quad N(\epsilon,D) \geqslant 1 \\ K\lambda_0 & , \text{ if } \quad N(\epsilon,D) = 0 \end{cases} . \tag{3.44}$$

□

In the proof of Lemma 3.2 and several results in the remainder of this chapter we use the following discrete-time version of the Gronwall inequality.

Lemma 3.3: If r(k), p(k) are sequences of nonnegative numbers satisfying

$$r(k) \leqslant K\lambda^{k-k_0}r(k_0) + \sum_{i=k_0}^{k-1} K\lambda^{k-1-i}\rho(i)r(i) , \tag{3.45}$$

then

$$r(k) \leqslant Kr(k_0)\prod_{i=k_0}^{k-1}[\lambda+K\rho(i)] \quad . \tag{3.46}$$

Proof: Letting $r_1(k) = \lambda^{-k}r(k)$, we have

$$r_1(k) \leqslant Kr_1(k_0) + \sum_{i=k_0}^{k-1}\frac{K}{\lambda}\rho(i)r_1(i) \quad . \tag{3.47}$$

Letting $r_2(k_0) = Kr_1(k_0)$ and

$$r_2(k) = Kr_2(k_0) + \sum_{i=k_0}^{k-1}\frac{K}{\lambda}\rho(i)r_2(i) \quad , \tag{3.48}$$

we see that $r_2(k)$ satisfies the scalar ordinary difference equation

$$\begin{aligned} r_2(k+1) &= [1+\frac{K}{\lambda}\rho(k)]r_2(k) \\ &= \left\{\prod_{i=k_0}^{k}[1+\frac{K}{\lambda}\rho(i)]\right\}r_2(k_0) \quad . \end{aligned} \tag{3.49}$$

Comparing (3.47) and (3.48) it is clear that $r_1(k) \leqslant r_2(k)$; hence,

$$\begin{aligned} r(k) = \lambda^k r_1(k) &\leqslant r_2(k_0)\lambda^k\prod_{i=k_0}^{k-1}[1\frac{K}{\lambda}\rho(i)] \\ &= Kr(k_0)\lambda^{k-k_0}\prod_{i=k_0}^{k-1}[1+\frac{K}{\lambda}\rho(i)] \\ &= Kr(k_0)\prod_{i=k_0}^{k-1}[\lambda+K\rho(i)] \quad . \end{aligned} \tag{3.50}$$

□

Proof of Lemma 3.2: Letting $A(i)$ denote $A(p(\theta_j(i)))$, it follows from Assumption 3.3 and (3.41) that

$$|A(n_1)-A(n_2)| \leqslant \epsilon a\rho_F(D)|n_1-n_2| \quad . \tag{3.51}$$

In order to prove (3.43) it is sufficient to show that the solution of (3.39) satisfies

$$|z(n_1)| \leqslant K|z(n_2)|\lambda_1^{n_1-n_2}(\epsilon,D) \quad , \qquad n_1 \geqslant n_2 \tag{3.52}$$

for an arbitrary $z(n_2)$ and arbitrary n_2. For any integer u, we can rewrite (3.39) as

$$z(i+1) = A(u)z(i) + (A(i)-A(u))z(i) \quad . \tag{3.53}$$

Applying the variation of constants formula to (3.53) and taking norms we have

$$|z(n_1)| \leq K|z(n_2)|\lambda_0^{n_1-n_2} + \sum_{i=n_2}^{n_1-1} K\lambda_0^{n_1-1-i}\epsilon a\rho_F(D)|i-u| \; |z(i)| \quad . \tag{3.54}$$

which in light of Lemma 3.3 implies

$$|z(n_1)| \leq K|z(n_2)|\prod_{i=n_2}^{n_1-1}(\lambda_0+\epsilon Ka\rho_F(D)|i-u|) \quad . \tag{3.55}$$

For any given n_1 and n_2 we choose u as the integer $\frac{n_1-n_2}{2}$ or $\frac{n_1-n_2-1}{2}$; hence, $\sum_{i=n_2}^{n_1-1}|i-u| \leq \frac{(n_1-n_2)^2}{4}$. Noting that $\ln(x_1+x_2) \leq \ln(x_1)+\frac{x_2}{x_1}$ we have

$$\begin{aligned}
\prod_{i=n_2}^{n_1-1}[\lambda_0+\epsilon Ka\rho_F(D)|i-u|] &= \exp\left[\ln\prod_{i=n_2}^{n_1-1}[\lambda_0+\epsilon Ka\rho_F(D)|i-u|]\right] \\
&= \exp\left[\sum_{i=n_2}^{n_1-1}\ln[\lambda_0+\epsilon Ka\rho_F(D)|i-u|]\right] \\
&\leq \exp\left[\sum_{i=n_2}^{n_1-1}\ln[\lambda_0+\epsilon Ka\rho_F(D)\frac{n_1-n_2}{4}] + \sum_{i=n_2}^{n_1-1}\epsilon Ka\rho_F(D)[|i-u|-\frac{n_1-n_2}{4}]\right] \qquad (3.56) \\
&\leq \exp\left[\sum_{i=n_2}^{n_1-1}\ln[\lambda_0+\epsilon Ka\rho_F(D)\frac{n_1-n_2}{4}]\right] \\
&= [\lambda_0+\epsilon Ka\rho_F(D)\frac{n_1-n_2}{4}]^{n_1-n_2} \quad .
\end{aligned}$$

Thus, (3.55) is replaced by

$$|z(n_1)| \leq K|z(n_2)|[\lambda_0+\epsilon Ka\rho_F(D)\frac{n_1-n_2}{4}]^{n_1-n_2} \quad . \tag{3.57}$$

For $n_1-n_2 \leq N(\epsilon,D)$, we have

$$\lambda_0+\epsilon Ka\rho_F(D)\frac{n_1-n_2}{4} \leq \lambda_1(\epsilon,D) \; ; \tag{3.58}$$

hence, (3.43) holds for $n_1-n_2 \leq N(\epsilon,D)$. At $n_1=n_2+N(\epsilon,D)$ we have

$$\lambda_1^{N(\epsilon,D)}(\epsilon,D) \;=\; K[\lambda_0+\epsilon K a \rho_F(D)\frac{N(\epsilon,D)}{4}]^{N(\epsilon,D)} \quad . \tag{3.59}$$

which implies that for any integer $m \geq 0$

$$|z(n_2+mN(\epsilon,D))| \;\leq\; \lambda_1^{mN(\epsilon,D)}(\epsilon,D)|z(n_2)| \quad . \tag{3.60}$$

Since any interval of length n_1-n_2 can be broken up into an interval of length $mN(\epsilon,D)$ and one of length less than $N(\epsilon,D)$, this completes the proof. Note that the proof holds for any integer N in place of $N(\epsilon,D)$. The particular choice of $N(\epsilon,D)$ given by (3.42) approximately minimizes $\lambda_1(\epsilon,D)$.
□

Remark 3.2: $N(\epsilon,D) = 0$ is a degenerate case where $(K-1)\lambda_0 < \epsilon\frac{Ka\rho_F(D)}{4}$. When K is so close to 1 or λ_0 is so close to zero, we let $\lambda_0' = K\lambda_0$ and note from (3.9) that $|A(\theta)| \leq \lambda_0' \; \forall(\theta) \in \Theta$. Hence, solutions of (3.39) satisfy $|z(i)| \leq (\lambda_0')^{i-i_0}|z(i_0)|$ no matter how fast or slowly θ_j moves.
□

From the fact that $K^{(1/N)} \leq 1+\frac{K-1}{N}$, and from equations (3.42), and (3.44), we bound $\lambda_1(\epsilon,D)$ by

$$\lambda_1(\epsilon,D) \;\leq\; \left[1+\frac{\left[\frac{\epsilon a\rho_F(D)K(K-1)}{4\lambda_0}\right]^{1/2}}{1-\left[\frac{\epsilon a\rho_F(D)K}{4\lambda_0(K-1)}\right]^{1/2}}\right]\left[1+\left[\frac{\epsilon a\rho_F(D)K(K-1)}{4\lambda_0}\right]^{1/2}\right]\lambda_0 \quad . \tag{3.61}$$

Since $\lambda_0 < 1$ it is clear from (3.61) that ϵ can be chosen small enough so that $\lambda_1(\epsilon,D) < 1$; hence, for ϵ sufficiently small (3.39) is exponentially stable. We complete the construction of the map T_ϵ with the pointwise definition

$$(T_\epsilon H_j)(k,\theta) \;=\; \sum_{i=-\infty}^{k-1} \Phi_j(k-1,i;k,\theta,\epsilon)G(i+1,p(\theta_j(i;k,\theta,\epsilon)),H_j(i,\theta_j(i;k,\theta,\epsilon))) \quad . \tag{3.62}$$

Because $G(k,p(\theta),H_j(i,\theta))$ is uniformly bounded and Φ_j is exponentially decaying for ϵ sufficiently small, the right-hand side of (3.62) is bounded for each $k \in \mathbf{Z}$ and each $\theta \in R^{2p}$. If h is a fixed point of T_ϵ, then given any k_0, θ_0 choosing $z(k_0) = h(k_0,\theta_0;\epsilon)$ results in a solution $z(k)$, $\theta(k)$ such that $z(k) = h(k,\theta(k);\epsilon)$. Hence, the fixed point of T_ϵ is indeed the manifold function h which we are

seeking. We derive conditions for T_ϵ to be a contraction on $\mathbf{H}(D,\Delta)$ in the following lemma.

Lemma 3.4: Suppose that Assumptions 3.1-3.3 hold and that (3.41) holds. If $\epsilon>0$, $D>0$, and $\Delta>0$ satisfy

$$\epsilon\ \frac{Kv_1\rho_F(D)}{1-\lambda_1(\epsilon,D)} \leqslant D \ , \tag{3.63}$$

$$\lambda_1(\epsilon,D) < 1-\epsilon\rho_1(D,\Delta) \ , \tag{3.64}$$

$$\epsilon\ \frac{K}{1-\epsilon\rho_1(D,\Delta)-\lambda_1(\epsilon,D)}\left(\rho_2(D,\Delta)+\frac{Kav_1\rho_F(D)}{[1-\lambda_1(\epsilon,D)]}\right) \leqslant \Delta \ , \tag{3.65}$$

$$\epsilon\ \frac{\rho_z(D)}{1-\lambda_1(\epsilon,D)}(Kv_1+\Delta) < 1 \ , \tag{3.66}$$

where $\lambda_1(\epsilon,D)$ is given by (3.44), then T_ϵ is a contraction mapping on $\mathbf{H}(D,\Delta)$.

Proof: Let $H_j,H_m\in\mathbf{H}(D,\Delta)$ be arbitrary. The first bound (3.63) is the easiest to obtain. As $\epsilon v_1\rho_F(D)$ bounds $|G(i+1,p(\theta_j(i;k,\theta,\epsilon)),H_j(i,\theta_j(i;k,\theta,\epsilon)))|$, we have

$$|(T_\epsilon H_j)(k,\theta)| \leqslant \sum_{i=-\infty}^{k-1} K\lambda_1^{k-1-i}(\epsilon,D)\epsilon v_1\rho_F(D) = \epsilon\ \frac{Kv_1\rho_F(D)}{1-\lambda_1(\epsilon,D)} \ ; \tag{3.67}$$

hence, (3.63) ensures $\|T_\epsilon H_j\| \leqslant D$. The bound (3.64) arises in the derivation of (3.65) and (3.66) which guarantee that $|(T_\epsilon H_j)(k,\theta)-(T_\epsilon H_j)(k,\hat\theta)| \leqslant \Delta|\theta-\hat\theta|$ and $\|T_\epsilon H_j-T_\epsilon H_m\| < \|H_j-H_m\|$, respectively. The most difficult step in establishing (3.65) and (3.66) is determining the dependence of $\Phi_j(n_1,n_2;k,\theta,\epsilon)$ on θ and H_j. We do this now. Denoting $\Phi_j(n_1,n_2;k,\theta,\epsilon)$, $\Phi_m(n_1,n_2;k,\hat\theta,\epsilon)$, $\theta_j(i;k,\theta,\epsilon)$, and $\theta_m(i;k,\hat\theta,\epsilon)$ by $\Phi(n_1,n_2),\hat\Phi(n_1,n_2)$, $\theta(i)$, and $\hat\theta(i)$, respectively, we write the ordinary difference equation in n_1

$$\begin{aligned}\Phi(n_1+1,n_2)-\hat\Phi(n_1+1,n_2) &= A(p(\theta(n_1)))[\Phi(n_1,n_2)-\hat\Phi(n_1,n_2)] \\ &\quad + [A(p(\theta(n_1)))-A(p(\hat\theta(n_1)))]\hat\Phi(n_1,n_2) \ .\end{aligned} \tag{3.68}$$

Applying the variation of constants formula, we obtain

$$[\Phi(n_1,n_2)-\hat\Phi(n_1,n_2)] = [\Phi(n_2,n_2)-\hat\Phi(n_2,n_2)] + \sum_{i=n_2}^{n_1-1}\Phi(n_1-1,i)[A(p(\theta(i)))-A(p(\hat\theta(i)))]\hat\Phi(i,n_2) \tag{3.69}$$

for all $n_2 \leqslant n_1 \leqslant k$. Using $\Phi(n_2,n_2) = \hat{\Phi}(n_2,n_2) = I$, (3.41), (3.43) and the triangle inequality, we bound $\Phi(n_2,n_2) - \hat{\Phi}(n_1,n_2)$ by

$$|\Phi_j(n_1,n_2;k,\theta,\epsilon) - \Phi_m(n_1,n_2;k,\hat{\theta},\epsilon)| \leqslant K^2 a \lambda_1^{n_1-1-n_2}(\epsilon,D) \sum_{i=n_2}^{n_1-1} |\theta_j(i;k,\theta,\epsilon) - \theta_m(i;k,\hat{\theta},\epsilon)| \qquad (3.70)$$

Note that (3.64) implies $\epsilon < \epsilon_1(D,\Delta)$; hence, the bounds (3.33), (3.34) hold. Substituting from Lemma 3.1 we arrive at the key to (3.65),

$$\begin{aligned} &|\Phi_j(k-1,n;k,\theta,\epsilon) - \Phi_j(k-1,n;k,\hat{\theta},\epsilon)| \\ &\qquad \leqslant K^2 a \lambda_1^{k-2-n}(\epsilon,D)\, |\theta-\hat{\theta}| \sum_{i=n}^{k-2} \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)} \right]^{k-i} \\ &\qquad = K^2 a |\theta-\hat{\theta}| \lambda_1^{k-2-n}(\epsilon,D)\, \frac{1}{\epsilon\rho_1(D,\Delta)} \left| \left[\frac{1}{\epsilon\rho_1(D,\Delta)} \right]^{k-1-n} - 1 \right| \qquad (3.71) \\ &\qquad = \frac{K^2 a |\theta-\hat{\theta}|}{\epsilon\rho_1(D,\Delta)\lambda_1(\epsilon,D)} \left| \left[\frac{\lambda_1(\epsilon,D)}{1-\epsilon\rho_1(D,\Delta)} \right]^{k-1-n} - \lambda_1^{k-1-n}(\epsilon,D) \right| . \end{aligned}$$

and the key to (3.66),

$$\begin{aligned} &|\Phi_j(k-1,n;k,\theta,\epsilon) - \Phi_m(k-1,n;k,\theta,\epsilon)| \\ &\leqslant K^2 a \lambda_1^{k-i-2}(\epsilon,D) \|H_j - H_m\| \frac{\rho_2(D)}{\rho_1(D,\Delta)} \sum_{i=n}^{k-2} \left| \left[\frac{1}{1-\epsilon\rho_1(D,\Delta)} \right]^{k-i} - 1 \right| \qquad (3.72) \\ &= \frac{K^2 a \rho_2(D) \|H_j - H_m\|}{\epsilon\rho_1^2(D,\Delta)\lambda_1(\epsilon,D)} \left| \left[\frac{\lambda_1(\epsilon,D)}{1-\epsilon\rho_1(D,\Delta)} \right]^{k-1-n} - \lambda_1^{k-1-n}(\epsilon,D) - \epsilon\rho_1(D,\Delta)(k-1-n)\lambda_1^{k-1-n}(\epsilon,D) \right| . \end{aligned}$$

Using the inequality (3.71) we bound the dependence of $T_\epsilon H_j$ on θ. With the same notation as in (3.68) and $j=m$, the triangle inequality gives

$$
\begin{aligned}
&|(T_\epsilon H_j)(k,\theta)-(T_\epsilon H_j)(k,\hat{\theta})| \\
&\qquad \leqslant \sum_{i=-\infty}^{k-1} |\Phi(k-1,i)[G(i+1,p(\theta(i)),H_j(i,\theta(i)))-G(i+1,p(\hat{\theta}(i)),H_j(i,\hat{\theta}(i)))]| \\
&\qquad\quad + \sum_{i=-\infty}^{k-1} |[\Phi(k-1,i)-\hat{\Phi}(k-1,i)]G(i+1,p(\hat{\theta}(i)),H_j(i,\hat{\theta}(i)))| \\
&\qquad \leqslant \sum_{i=-\infty}^{k-1} \epsilon K \lambda_1^{k-i-1}(\epsilon,D)\rho_2(D,\Delta)\left|\frac{1}{1-\epsilon\rho_1(D,\Delta)}\right|^{k-i}|\theta-\hat{\theta}| \\
&\qquad\quad + \sum_{i=-\infty}^{k-1} \frac{\epsilon v_1\rho_F(D)K^2 a|\theta-\hat{\theta}|}{\epsilon\rho_1(D,\Delta)\lambda_1(\epsilon,D)}\left[\left|\frac{\lambda_1(\epsilon,D)}{1-\epsilon\rho_1(D,\Delta)}\right|^{k-i-1}-\lambda_1^{k-1-i}(\epsilon,D)\right] \\
&\qquad \leqslant \epsilon\,\frac{K}{1-\epsilon\rho_1(D,\Delta)-\lambda_1(\epsilon,D)}\left[\rho_2(D,\Delta)+\frac{v_1 K a\rho_F(D)}{[1-\lambda_1(\epsilon,D)]}\right]|\theta-\hat{\theta}| \ .
\end{aligned}
\tag{3.73}
$$

where the last line holds because of (3.64). Thus, (3.63)-(3.65) imply that $T_\epsilon H_j \in \mathbf{H}(D,\Delta)$ if $H_j \in \mathbf{H}(D,\Delta)$. Denoting $\Phi_j(k-1,i;k,\theta,\epsilon)$, $\theta_j(i;k,\theta,\epsilon)$ by $\Phi_j(k-1,i)$, $\theta_j(i)$, respectively, and similarly for Φ_m, ϑ_m, we have

$$
\begin{aligned}
&|(T_\epsilon H_j)(k,\theta)-(T_\epsilon H_m)(k,\theta)| \\
&\leqslant \sum_{i=-\infty}^{k-1} |\Phi_j(k-1,i)[G(i+1,p(\theta_j(i)),H_j(i,\theta_j(i)))-G(i+1,p(\theta_m(i)),H_j(i,\theta_m(i)))]| \\
&\quad + \sum_{i=-\infty}^{k-1} |\Phi_j(k-1,i)[G(i+1,p(\theta_m(i)),H_j(i,\theta(i)))-G(i+1,p(\theta_m(i)),H_m(i,\theta_m(i)))]| \\
&\quad + \sum_{i=-\infty}^{k-1} |[\Phi_j(k-1,i)-\Phi_m(k-1,i)]G(i+1,p(\theta_m(i)),H_m(i,\theta_m(i)))| \\
&\leqslant \sum_{i=-\infty}^{k-1} \frac{\epsilon K\rho_2(D,\Delta)\rho_2(D)\|H_j-H_m\|}{\rho_1(D,\Delta)[1-\epsilon\rho_1(D,\Delta)]}\left[\left(\frac{\lambda_1(\epsilon,D)}{1-\epsilon\rho_1(D,\Delta)}\right|^{k-1-i}-[1-\epsilon\rho(D,\Delta)]\lambda_1^{k-1-i}(\epsilon,D)\right] \\
&\quad + \sum_{i=-\infty}^{k-1} \epsilon K v_1\rho_2(D)\|H_j-H_m\|\lambda_1^{k-1-i}(\epsilon,D) \\
&\quad + \sum_{i=-\infty}^{k-1} \frac{\epsilon v_1\rho_F(D)K^2 a\rho_2(D)\|H_j-H_m\|}{\epsilon\rho_1^2(D,\Delta)\lambda_1(\epsilon,D)}\left[\left|\frac{\lambda_1(\epsilon,D)}{1-\epsilon\rho_1(D,\Delta)}\right|^{k-1-i}-\lambda_1^{k-1-i}(\epsilon,D)\right. \\
&\qquad\qquad \left. -\,\epsilon\rho_1(D,\Delta)(k-1-i)\lambda_1^{k-1-i}(\epsilon,D)\right] \\
&= \frac{\epsilon\rho_2(D)\|H_j-H_m\|}{1-\lambda_1(\epsilon,D)}\left[Kv_1+\frac{\epsilon K}{1-\epsilon\rho_1(D,\Delta)-\lambda_1(\epsilon,D)}\left[\rho_2(D,\Delta)+\frac{v_1 K a\rho_F(D)}{[1-\lambda_1(\epsilon,D)]}\right]\right] \\
&\leqslant \frac{\epsilon\rho_2(D)}{1-\lambda_1(\epsilon,D)}(Kv_1+\Delta)\|H_j-H_m\| \ .
\end{aligned}
\tag{3.74}
$$

□

Remark 3.3: Lemma 3.4 uses $\lambda_1(\epsilon, D)$ given by (3.44) only to imply that the stability bound (3.43) holds. If (3.43) can be established with λ_1 replaced by λ_1', then λ_1' can be used in (3.63)-(3.66). For example, if there exists a positive definite matrix P such that

$$A^T(\theta)PA(\theta) \leqslant \lambda_0 P \quad \forall \theta \in \Theta \ . \tag{3.75}$$

then (3.43) is satisfied with $\lambda_1(\epsilon, D) = \lambda_0$ and $K = (\lambda_{max}P/\lambda_{min}P)^{1/2}$.

□

In light of (3.61), the fact that $\lambda_0 < 1$, and the fact that $\lambda_1(\epsilon, D)$ is a nondecreasing function of ϵ and D, it is clear that for any fixed $D_0 > 0$, $\Delta_0 > 0$ we can find an $\epsilon_2(D_0, \Delta_0)$ such that (3.63)-(3.66) hold for all $\epsilon \in [0, \epsilon_2)$. Because ρ_F, ρ_z, ρ_1, and ρ_2 are nondecreasing functions of D, Δ, it follows that (3.63)-(3.66) hold with $D = (\epsilon/\epsilon_2)D_0$ and $\Delta = (\epsilon/\epsilon_2)\Delta_0$ for all $\epsilon \in [0, \epsilon_2)$. We illustrate the last statement with (3.63),

$$\epsilon \ \frac{Kv_1\rho_F((\epsilon/\epsilon_2)D_0)}{1-\lambda_1(\epsilon,(\epsilon/\epsilon_2)D_0)} \leqslant \epsilon \ \frac{Kv_1\rho_F(D_0)}{1-\lambda_1(\epsilon,D_0)} \leqslant \epsilon \ \frac{D_0}{\epsilon_2} \ . \tag{3.76}$$

By the Banach fixed point theorem, there exists a unique function $h(\cdot,\cdot;\epsilon) \in \mathbf{H}((\epsilon/\epsilon_2)D_0,(\epsilon/\epsilon_2)\Delta_0)$ which is the fixed point of T_ϵ, and, hence, defines via (3.16) the integral manifold M_ϵ of (3.27)-(3.28) for all $\epsilon \in [0, \epsilon_2)$. This result is summarized in the following theorem.

Theorem 3.1: Suppose that Assumptions 3.1-3.3 hold and that (3.41) holds. Given any fixed $D_0 > 0$, $\Delta_0 > 0$, there exists $\epsilon_2(D_0, \Delta_0) > 0$ such that for each $\epsilon \in [0, \epsilon_2)$ the modified system (3.27)-(3.28) possesses an integral manifold M_ϵ defined by (3.16) with $h(\cdot,\cdot;\epsilon) \in \mathbf{H}((\epsilon/\epsilon_2)D_0,(\epsilon/\epsilon_2)\Delta_0)$.

□

This result translates to the original system (3.1)-(3.2) as follows.

Corollary 3.1: Under the conditions of Theorem 3.1, suppose that $\epsilon < \epsilon_2(D_0, \Delta_0)$. Let $x(k)$, $\theta(k)$ be the solution of (3.1)-(3.2) with initial data $x(k_0) = x_0$, $\theta(k_0) = \theta_0$. Suppose that $\theta(k) \in \Theta_1(\epsilon,(\epsilon/\epsilon_2)D_0)$ for all $k \in [k_0, k_1]$. If $(k_0, \theta_0, x_0) \in M_\epsilon$, then $(k, \theta(k), x(k)) \in M_\epsilon$ for all $k \in [k_0, k_1]$ where M_ϵ is defined by (3.3).

□

Remark 3.4: Up to this point the only assumption we have made about the external inputs to the system, $w(k)$ and $f(k,\cdot,\cdot)$, is that they are uniformly bounded. If they are periodic (almost periodic), then $\nu(k,\theta)$ and $h(k,\theta;\epsilon)$ are periodic (almost periodic) in k.

□

Before considering the attractivity of M_ϵ in the next section, we give an instability result. The evolution of (3.27)-(3.28) restricted to M_ϵ is governed by the reduced-order system

$$\theta(k+1) = \theta(k) + \epsilon F(k,p(\theta(k)),h(k,\theta(k);\epsilon)) \quad . \tag{3.77}$$

The proof of the following theorem is identical to the proof of Theorem 2.2.

Theorem 3.2: Under the conditions of Theorem 3.1, suppose that $\epsilon<\epsilon_2(D_0,\Delta_0)$. Let $\theta^*(k)$ be a solution of (3.77). If $\theta^*(k)$ is an unstable solution of (3.77), then $z^* = h(k,\theta^*(k);\epsilon)$, $\theta^*(k)$ is an unstable solution of (3.27)-(3.28).

□

3.4. Attractivity of the Slow Manifold

Theorem 3.2 showed that the existence of M_ϵ is sufficient to prove that instability in the reduced-order system (3.77) implies instability in the full-order system (3.27)-(3.28). However, existence of M_ϵ is not sufficient to show that the existence of a stable solution of (3.77) implies the existence of a stable solution of (3.27)-(3.28). In this section we establish the exponential attractivity of M_ϵ which is sufficient to show that the existence of a uniformly (asymptotically) stable solution of (3.77) implies the existence of a uniformly (asymptotically) stable solution of (3.27)-(3.28).

We begin with a boundedness result for $z(k)$ which justifies Remark 2.8.

Lemma 3.5: Suppose that Assumptions 3.1-3.3 hold and that (3.41) holds. Given any fixed $\lambda \in (\lambda_0,1)$, $D_0 > 0$, and $D_1 \geqslant D_0$, let $p(\theta)$ take values in $\Theta_1(\epsilon,D_1)$. There exists $\epsilon_3(D_0,D_1,\lambda) > 0$ such that for each $\epsilon \in [0,\epsilon_3]$ if $|z(k_0)| \leqslant D_1/K$ and $|\theta(k_0)| < \infty$, then

$$|z(k)| \leqslant K\lambda^{k-k_0}|z(k_0)| + (\epsilon/\epsilon_3)D_0(1-\lambda^{k-k_0}) \leqslant D_1 . \tag{3.78}$$

Proof: Choose $\epsilon_3(D_0,D_1,\lambda_1)$ so that

$$\lambda_1(\epsilon_3,D_1) \leqslant \lambda_1 , \quad \epsilon_3 K v_1 \rho_F(D_1) \leqslant D_0(1-\lambda_1(\epsilon_3,D_1)) . \tag{3.79}$$

By (3.61) it is clear that such an $\epsilon_3 > 0$ exists. We prove (3.78) by induction. Suppose that $|z(i)| \leqslant D_1$ for all $i \in [k_0,k-1]$. Then $|F(i,p(\theta(i)),z(i))| \leqslant \rho_F(D_1)$ for all $i \in [k_0,k-1]$; hence, $|\theta(i)| < \infty$ for all $i \in [k_0,k-1]$. Letting Φ be the state transition matrix of $z(i+1) = A(\theta(i))z(i)$ it follows from Lemma 3.2 that $|\Phi(n_1,n_2)| \leqslant K\lambda_1^{n_1-n_2}(\epsilon,D_1)$ for all $n_1,n_2 \in [k_0,k]$. Applying the variation of constants formula to (3.27), we have

$$z(k) = \Phi(k,k_0)z(k_0) - \sum_{i=k_0}^{k-1} \Phi(k-1,i)G(i+1,p(\theta(i)),z(i)) . \tag{3.80}$$

Because $|z(i)| \leqslant D_1$ implies that $|G(i+1,\theta(i),z(i))| \leqslant \epsilon v_1 \rho_F(D_1)$, it follows from (3.80) that

$$|z(k)| \leqslant K\lambda_1^{k-k_0}(\epsilon,D_1)\,|z(k_0)| + \epsilon\frac{Kv_1\rho_F(D_1)}{1-\lambda_1(\epsilon,D_1)}[1-\lambda_1^{k-k_0}(\epsilon,D_1)] . \tag{3.81}$$

which implies for $\epsilon \in [0,\epsilon_2]$ that $|z(k)| \leqslant D_1$. Hence, if (3.78) holds for all $i\in[k_0,k]$, then it holds for all $i\in[k_0,k+1]$. Since $|z(k_0)| \leqslant D_1$, (3.78) holds for all $k \geqslant k_0$.

□

Remark 3.5: For Lemma 3.5 it is sufficient that $\nu(k,\theta)$ be Lipschitzian in θ and that $F(k,\theta,z)$ be bounded for all $k \in [k_0,\infty)$, $\theta \in \Theta$, and z in compact sets. That is, the assumption of a Lipschitzian derivative of ν and the assumption that $F(k,\theta,z)$ is Lipschitzian in θ, z can be dropped from Assumptions 3.2 and 3.3, respectively.

□

Although Lemma 3.5 shows that $z(k)$ converges exponentially to a ball with radius $O(\epsilon)$, it does not show that M_ϵ is exponentially attractive. In order to establish the exponential attractivity of M_ϵ we introduce the deviation of z from $h(k,\theta;\epsilon)$ as a new state variable

$$\eta = z - h(k,\theta;\epsilon) . \tag{3.82}$$

Using the fact that h satisfies the functional difference equation

$$h(k+1,\theta + \epsilon F(k,p(\theta),h(k,\theta;\epsilon));\epsilon) = A(p(\theta))h(k,\theta;\epsilon) - G(k+1,p(\theta),h(k,\theta;\epsilon)), \tag{3.83}$$

we obtain the system of ordinary difference equations for (3.27)-(3.28) in η, θ coordinates

$$\eta(k+1) = A(p(\theta(k)))\eta(k) - G'(k+1,\theta(k),\eta(k);\epsilon) \tag{3.84}$$

$$\theta(k+1) = \theta(k) + \epsilon F'(k,\theta(k),\eta(k);\epsilon), \tag{3.85}$$

where

$$\begin{aligned} G'(k+1,\theta,\eta;\epsilon) &= \nu(k+1,p(\theta) + \epsilon F'(k,\theta,\eta)\epsilon) - \nu(k+1,p(\theta) + \epsilon F'(k,\theta,0)\epsilon) \\ &\quad + h(k+1,p(\theta) + \epsilon F'(k,\theta,\eta;\epsilon);\epsilon) - h(k+1,p(\theta) + \epsilon F'(k,\theta,0);\epsilon), \end{aligned} \tag{3.86}$$

$$F'(k,\theta,\eta;\epsilon) = F(k,p(\theta),h(k,\theta;\epsilon) + \eta). \tag{3.87}$$

Remark 3.6: We could not use $\eta = z-h$ as a state variable in the continuous-time case in Chapter 2 because we did not prove that $\frac{\partial h}{\partial \theta}$ existed and $\dot{\eta} = \dot{z} - \frac{\partial h}{\partial t} - \frac{\partial h}{\partial \theta}\dot{\theta}$.

□

With the help of Lemma 3.5 it is straightforward to show that η converges exponentially to zero. We summarize the existence and attractivity results in the following theorem.

Theorem 3.3: Suppose that Assumptions 3.1-3.3 hold and that (3.41) holds. Given any fixed $\lambda \in (\lambda_0,1)$, $D_0 > 0$, $\Delta_0 > 0$, and $D_1 \geq D_0$, let $p(\theta)$ take values in $\Theta_1(\epsilon,D_1)$. There exists $\epsilon_4(D_0,\Delta_0,D_1,\lambda) > 0$ such that for each $\epsilon \in [0,\epsilon_4)$ (3.27)-(3.28) possesses an integral manifold M_ϵ given by (3.16) with $h(\cdot,\cdot;\epsilon) \in \mathbf{H}((\epsilon/\epsilon_2)D_0, (\epsilon/\epsilon_2)\Delta_0)$, where $\epsilon_2 = \epsilon_2(D_0,\Delta_0) \geq \epsilon_4$ is from Theorem 3.1. Furthermore, if $z(k)$, $\theta(k)$ is the solution of (3.27)-(3.28) with initial data $\theta(k_0) = \theta_0 \in R^{n_\theta}$, $z(k_0) = z_0$, and if $|z_0| \leq D_1/K$, then $z(k)$ satisfies (3.78) and $\eta(k) = z(k) - h(k,\theta(k);\epsilon)$ satisfies

$$|\eta(k)| \leq K\lambda^{k-k_0}|\eta(k_0)|. \tag{3.88}$$

Proof: Choose $\epsilon_4(D_0,\Delta_0,D_1,\lambda) \leq \min\{\epsilon_2(D_0,\Delta_0), \epsilon_3(D_0,D_1,\lambda)\}$ such that

$$\lambda_1(\epsilon_4,D_1) + \epsilon_4 K\rho_z(D_1)[v_1 + (\epsilon_4/\epsilon_2)\Delta_0] \leq \lambda. \tag{3.89}$$

The existence of M_ϵ follows from $\epsilon_4 \leq \epsilon_2$. That $z(k)$ satisfies (3.78) follows from $\epsilon_4 \leq \epsilon_3$. With

$|z| = |h + \eta| \leqslant D_1$, we bound G' by

$$|G'(k+1,\theta,\eta;\epsilon)| \leqslant \epsilon\,\rho_2(D_1)[v_1 + (\epsilon/\epsilon_2)\Delta_0]\,|\eta|\,. \tag{3.90}$$

Applying the variation of constants formula to (3.84), we have

$$\eta(k) = \Phi(k,k_0)\,\eta(k_0) - \sum_{i=k_0}^{k-1} \Phi(k-1,i)G'(i+1,\theta(i),\eta(i);\epsilon)\,, \tag{3.91}$$

where Φ is the same state transition matrix as in Lemma 3.5. Taking norms and applying Lemma 3.3 give

$$|\eta(k)| \leqslant K[\lambda_1(\epsilon,D_1) + \epsilon\,K\rho_2(D_1)(v_1 + (\epsilon/\epsilon_2)\Delta_0)]^{k-k_0}\,|\eta(k_0)|\,, \tag{3.92}$$

which, in light of (3.89), completes the proof.

□

We illustrate the use of Theorem 3.3 in an example. First we rewrite (3.85) as a perturbed version of (3.77)

$$\begin{aligned}\theta(k+1) = \theta(k) &+ \epsilon F(k,p(\theta(k)),h(k,\theta(k);\epsilon)) \\ &+ \epsilon[F'(k,\theta(k),\eta(k);\epsilon) - F'(k,\theta(k),0;\epsilon)]\end{aligned} \tag{3.93}$$

where the perturbation is exponentially decaying to zero. For simplicity we consider the case where $\theta = 0$ is an equilibrium of (3.77), $\mathbf{B}(C,0) \subseteq \Theta$, and for each $\epsilon \in [0,\epsilon_4(D_0,\Delta_0,D_1,\lambda))$ we have

$$|\theta + \epsilon F(k,\theta,h(k,\theta;\epsilon))| \leqslant (1-\epsilon\lambda_2)\,|\theta|\,, \quad \forall\,\theta \in \mathbf{B}(C,0)\,, \tag{3.94}$$

where $\lambda_2 \in [0,\rho_1(0,0))$. Note that this implies $|p(\theta) + \epsilon F(k,p(\theta),h(k,\theta;\epsilon))| \leqslant |p(\theta)|$; hence, we can take $\Theta_1(\epsilon,D)=\mathbf{B}(C,0)$. Note, also that (3.94) is almost never satisfied with $\lambda_2 > 0$ in adaptive systems of the type (3.1)-(3.2). However, the basic idea does not change if the right-hand side of (3.94) is changed to $(1-\epsilon\lambda_2)|\theta|+\epsilon\delta$ where $\delta\epsilon[0,\lambda_2 C)$.

Because (3.94) implies that $\theta = 0$ is a uniformly stable or an exponentially stable equilibrium of (3.77) and because the perturbation in (3.93) is exponentially decaying to zero, it follows that $\theta = 0$ is a uniformly stable or an exponentially stable, respectively, equilibrium of (3.93). Therefore, the solution $\eta = 0$, $\theta = 0$ of (3.84)-(3.85) is uniformly stable if $\lambda_2 = 0$ and

exponentially stable if $\lambda_2 > 0$. The interesting problem is to estimate the region of attraction of this solution. Letting $r_\eta(k)$ satisfy

$$r_\eta(k+1) = \lambda r_\eta(k)\,, \qquad r(k_0) = K\,|\eta(k_0)|\,, \tag{3.95}$$

it is clear that $|\eta(k)| \leqslant r_\eta(k)$ for all $k \geqslant k_0$ for each $\epsilon \in [0,\epsilon_4(D_0,\Delta_0,D_1,\lambda))$. Letting $r_\theta(k)$ satisfy

$$r_\theta(k+1) = (1-\epsilon\lambda_2)\,r_\theta(k) + \epsilon\rho_2(D_1)\,r_\eta(k)\,, \qquad r_\theta(k_0) = |\theta(k_0)|\,, \tag{3.96}$$

it follows that if $\theta(i) \in \mathbf{B}(C,0)$ for all $i \in [k_0,k-1]$, then $\theta(k) \in \mathbf{B}(C,0)$. Thus, we can estimate the region of attraction by finding conditions such that the solution of the linear time-invariant system (3.95)-(3.96) keeps $r_\theta(k) \leqslant C$ for all $k \geqslant k_0$. Taking

$$V(r_\theta,r_\eta;\epsilon) = r_\theta + \frac{\epsilon\rho_2(D_1)}{1-\epsilon\lambda_2-\lambda}\,r_\eta \tag{3.97}$$

we have

$$\begin{aligned} V(r_\theta(k+1),r_\eta(k+1);\epsilon) &= (1-\epsilon\lambda_2)\,V(r_\theta(k),\,r_\eta(k);\epsilon) \\ &= (1-\epsilon\lambda_2)^{k-k_0}\,V(r_\theta(k_0),\,r_\eta(k_0);\epsilon)\,. \end{aligned} \tag{3.98}$$

which proves the following corollary. Because we are proving $\theta \in \Theta$ for all $k \geqslant k_0$, we can state the result directly for the original system (3.1)-(3.2).

Corollary 3.2: Suppose that Assumptions 3.1-3.3, (3.41), and (3.94) hold, that $\epsilon < \epsilon_4(D_0,\Delta_0,D_1,\lambda)$ and that $\mathbf{B}(C,0) \subseteq \Theta$. Let $x(k)$, $\theta(k)$ be the solution of (3.1)-(3.2) with initial data $x(k_0) = x_0$, $\theta(k_0) = \theta_0$. If $|x_0-\nu(k_0,\theta_0)| \leqslant D_1$ and

$$V_0 = V(|\theta_0|\,,\,K\,|x_0 - g(k_0,\theta_0;\epsilon)|\,;\epsilon) \leqslant C\,, \tag{3.99}$$

then

$$|x(k)-g(k,\theta(k);\epsilon)| \leqslant K\lambda^{k-k_0}\,|x_0-g(k_0,\theta_0;\epsilon)|\,, \qquad |\theta(k)| \leqslant KV_0(1-\epsilon\lambda_2)^{k-k_0}\,. \tag{3.100}$$

□

In many algorithms the dependence of $f(k,\theta,x)$ on x is quadratic or higher order. Then there exist $\rho_3(D_0)$ and $\rho_4(D_1)$ such that

$$\rho_z(|z|) \leqslant \rho_3(D_0) + \rho_4(D_1)|\eta| . \tag{3.101}$$

In such cases we can arrive at a less conservative estimate of the region of attraction.

Corollary 3.3: In Corollary 3.2, if (3.101) holds, then $V(r_\theta,r_\eta;\epsilon)$ given by (3.97) can be replaced by

$$V(r_\theta,r_\eta;\epsilon) = r_\theta + \epsilon \frac{\rho_3(D_0)}{1-\epsilon\lambda_2-\lambda} r_\eta + \epsilon \frac{\rho_4(D_1)}{1-\epsilon\lambda_2-\lambda^2} r_\eta^2 . \tag{3.102}$$

□

Thus, we have shown that the possession of an exponentially attractive integral manifold M_ϵ by (3.27)-(3.28) implies that the full-order system has the property that the existence of a uniformly (asymptotically) stable, an exponentially stable, or an unstable solution of the reduced-order system in M_ϵ (3.77) implies the existence of a uniformly (asymptotically) stable, an exponentially stable, or an unstable solution of the full-order system (3.27)-(3.28), respectively. Furthermore, if this solution of (3.77) lies in the interior of Θ, then the corresponding solution of the modified system (3.27)-(3.28) is transformed via $x(k) = \nu(k,\theta(k)) + z(k)$ into a solution of the original system (3.1)-(3.2). In the next section we study via averaging the behavior of solutions of (3.77). We conclude this section with a reminder that the assumptions under which these results were derived are very mild. This is especially true on the inputs to the system, namely, $w(k)$ and the k dependence of $f(k,\theta,x)$, which are only required to be uniformly bounded.

3.5. Analysis in the Manifold: Averaging

The system (3.1)-(3.2) restricted to M_ϵ behaves according to (3.4) which is in the standard Bogoliubov form for the method of averaging. However, the literature on the method of averaging for discrete-time systems with deterministic inputs is almost nonexistent. Meerkov (1973) presents elegant proofs using simple mathematics of several standard averaging theorems for continuous-time systems. He then states the corresponding theorems for discrete-time systems pointing out that the discrete-time proof, which is not given, is virtually a copy of the continuous-time proof. As the hypotheses of the theorems provided by Meerkov are somewhat different than

the hypotheses which our system satisfies. we shall state and prove several basic theorems from the method of averaging. Our proofs are modelled on Meerkov's but our theorem statements are in the style of Sethna and Moran (1968). While the literature on the method of averaging for discrete-time systems with deterministic inputs is scarce, there are many references which relate the behavior of (3.4) or (3.1)-(3.2) with stochastic inputs to the behavior of the ODE (3.5): Derevitskii and Fradkov (1974), Ljung (1977), Kushner (1977), Kushner and Clark (1978), Benveniste, Goursat, and Ruget (1980), Benveniste, Ruget (1982), Kushner and Swartz (1984), Metivier and Priouret (1984), just to mention a few. We conclude this section by showing that some with probability one results relating the behavior of (3.1)-(3.2) to that of the ODE (3.5) can be stated as corollaries to our basic averaging theorems. We feel that our approach of giving a complete deterministic proof and then adding stochastic assumptions offers the simplest introduction to this area, and is at least of pedestrian interest.

In order to simplify the appearance of the equations, we shall make a few notational changes. We assume that constants $D_0 > 0$, $\Delta_0 > 0$, $D_1 \geqslant D_0$, and $\lambda \in (\lambda_0, 1)$ have been chosen. We let $\epsilon_4 = \epsilon_4(D_0, \Delta_0, D_1, \lambda)$, $\rho_F = \rho_F((\epsilon_4/\epsilon_2)\, D_0)$, and $\rho_1 = \rho_1((\epsilon_4/\epsilon_2)\, D_0,\ (\epsilon_4/\epsilon_2)\Delta_0)$ where $\epsilon_2 = \epsilon_2(D_0, \Delta_0)$. Finally, we use $f(k,\theta;\epsilon)$ to denote $f(k,\theta,g(k,\theta;\epsilon)) = F(k,\theta,h(k,\theta;\epsilon))$ and we define $\rho_\epsilon = (1/\epsilon_2)\, D_0 \rho_z((\epsilon_4/\epsilon_2)\, D_0)$. It follows that for each $\epsilon \in [0,\epsilon_4)$, every $\theta, \theta' \in \Theta$, and all $k \in \mathbf{Z}$

$$|f(k,\theta;\epsilon)| \leqslant \rho_F, \quad |f(k,\theta,\epsilon) - f(k,\theta';\epsilon)| \leqslant \rho_1 |\theta - \theta'|, \quad |f(k,\theta;\epsilon) - f(k,\theta;0)| \leqslant \epsilon \rho_\epsilon . \tag{3.103}$$

As we are interested in the behavior of solutions of (3.4) or (3.77) only for $\theta \in \Theta$, we do not need the projection $p(\theta)$; hence, we study

$$\theta(k+1) = \theta(k) + \epsilon f(k,\theta;\epsilon) . \tag{3.104}$$

We use the classical notation $\theta(k;\theta_0,k_0)$ to represent the solution of (3.104) with initial data $\theta(k_0) = \theta_0$.

Assumption 3.4: The limit on the right-hand side of the definition

$$\overline{f}(\theta) \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{i=k}^{k+N-1} f(i,\theta;0) \tag{3.105}$$

exists uniformly with respect to $k \in \mathbf{Z}$ and $\theta \in \Theta$. That is, there exists a strictly increasing continuous function $\kappa(\cdot)$ with $\kappa(0) = 0$ such that

$$\left| \overline{f}(\theta) - \frac{1}{N} \sum_{i=k}^{k+N-1} f(i,\theta;0) \right| < \kappa\left(\frac{1}{N}\right) \tag{3.106}$$

for all $k \in \mathbf{Z}$ and $\theta \in \Theta$.

□

Remark 3.7: If $f(k,\theta;0)$ is N-periodic then we define $\overline{f}$ by

$$\overline{f}(\theta) = \frac{1}{N} \sum_{i=0}^{N-1} f(i,\theta;0) \tag{3.107}$$

and we take $\kappa(\frac{1}{N}) = 0$ in the following derivations.

□

We relate solutions of (3.104) to solutions of the ODE

$$\frac{d}{d\tau}\overline{\theta} = \overline{f}(\overline{\theta}). \tag{3.108}$$

We use the notation $\overline{\theta}(\tau;\theta_0)$ to represent the solution of (3.108) with initial data $\overline{\theta}(0) = \theta_0$. (Since (3.108) is time-invariant, there is no loss of generality in taking τ=0 as the initial time in (3.108).) Because $\overline{f}$ represents the average of f, (3.108) is also referred to as the <u>averaged system</u>. Our tool for establishing relationships between the trajectory of (3.104), $\theta(k;\theta_0,k_0)$, and the trajectory of (3.108) sampled at $t_k = \epsilon(k-k_0)$, $\overline{\theta}(t_k;\theta_0)$, is the <u>averaged trajectory</u> defined by

$$\hat{\theta}(k;\theta_0,k_0) = \frac{1}{N} \sum_{i=k}^{k+N-1} \theta(i;\theta_0,k_0). \tag{3.109}$$

where N is to be determined and is possibly a function of ϵ. The averaged trajectory $\hat{\theta}(k;\theta_0,k_0)$ is simply a moving average of length N of the trajectory $\theta(i;\theta_0,k_0)$ over a window beginning at $i = k$. We make use of the average trajectory in a two-step procedure. First, we bound the distance between $\theta(k;\theta_0,k_0)$ and $\hat{\theta}(k;\theta_0,k_0)$ and second, we bound the distance between $\hat{\theta}(k;\theta_0,k_0)$ and

$\bar{\theta}(t_k;\theta_0)$. Then the triangle inequality gives us a bound on the distance between $\theta(k;\theta_0,k_0)$ and $\bar{\theta}(t_k;\theta_0)$.

Lemma 3.6: Suppose that $\theta(k;\theta_0,k_0) \in \Theta$ for all $k \in [k_0,k_1+N-1]$. Then, for each $\epsilon \in (0,\epsilon_4)$ the distance between the original and averaged trajectories is bounded by

$$|\theta(k;\theta_0,k_0) - \hat{\theta}(k;\theta_0,k_0)| \leqslant (\epsilon N-\epsilon)(\rho_F/2) \tag{3.110}$$

for all $k \in [k_0,k_1]$.

Proof: Using (3.103) we have the bound for $i \in [k_0,k_i+N-1]$

$$|\theta(i+1;\theta_0,k_0) - \theta(i;\theta_0,k_0)| \leqslant \epsilon\rho_F . \tag{3.111}$$

which implies that for $k \in [k_0,k_1]$ and $i \in [k,k_1+N-1]$

$$|\theta(i;\theta_0,k_0) - \theta(k;\theta_0,k_0)| \leqslant \epsilon\rho_F(i-k) . \tag{3.112}$$

From the definition of $\hat{\theta}$ and the triangle inequality we get

$$\begin{aligned} |\hat{\theta}(k;\theta_0,k_0) - \theta(k;\theta_0,k_0)| &\leqslant \frac{1}{N}\sum_{i=k}^{k+N-1} |\theta(j;\theta_0,k_0) - \theta(k;\theta_0,k_0)| \\ &\leqslant \frac{1}{N}\sum_{i=k}^{k+N-1} \epsilon\rho_F(i-k) \\ &= \frac{\epsilon\rho_F}{N}\,\frac{N(N-1)}{2} = (\epsilon N-\epsilon)(\rho_F/2) \end{aligned} \tag{3.113}$$

for all $k \in [k_0,k_1]$.

□

Lemma 3.7: Suppose that $\bar{\theta}(\tau;\theta_0) \in \Theta$ for all $\tau \in [0,\tau_1]$. Let $k_1(\epsilon) = [\tau_1/\epsilon]$, that is, the largest integer less than or equal to τ_1/ϵ. Then, for each $\epsilon \in (0,\epsilon_4)$ the trajectory $\bar{\theta}(\tau;\theta_0)$ of the ODE (3.108) sampled at $t_k = \epsilon(k-k_0)$ satisfies the ordinary difference equation

$$\bar{\theta}(t_{k+1}) = \bar{\theta}(t_k) + \epsilon\bar{f}(\bar{\theta}(t_k)) + f_1(\bar{\theta}(t_k);\epsilon) \tag{3.114}$$

for all $k \in [k_0,k_0+k_1(\epsilon)-1]$. Furthermore, f_1 defined by

$$f_1(\theta;\epsilon) = \bar{\theta}(\epsilon;\theta) - (\theta + \epsilon\bar{f}(\theta)) \tag{3.115}$$

satisfies

$$|f_1(\overline{\theta}(t_k;\theta_0);\epsilon)| \leq \epsilon^2(\rho_1\rho_F/2) \tag{3.116}$$

for all $k \in [k_0,k_0+k_1(\epsilon)-1]$.

Proof: It is clear from the time-invariant nature of (3.108) and $t_{k+1}-t_k = \epsilon$ that $\overline{\theta}(t_{k+1};\theta_0) = \overline{\theta}(\epsilon;\overline{\theta}(t_k;\theta_0))$; hence, (3.114) follows from the definition (3.115). To establish (3.116) we let θ denote $\overline{\theta}(t_k;\theta_0)$ for any $k \in [k_0,k_0+k_1(\epsilon)-1]$ and compute the bound

$$\begin{aligned} |\overline{\theta}(\epsilon;\theta)-\theta-\epsilon\overline{f}(\theta)| &= |\int_0^\epsilon (\overline{f}(\overline{\theta}(\tau;\theta)) - \overline{f}(\theta))d\tau| \\ &\leq \int_0^\epsilon \rho_1 |\overline{\theta}(\tau;\theta) - \theta| d\tau \\ &= \int_0^\epsilon \rho_1 |\theta + \int_0^\tau \overline{f}(\overline{\theta}(s;\theta))ds - \theta| d\tau \\ &\leq \rho_1\rho_F \int_0^\epsilon \int_0^\tau ds\, d\tau = \epsilon^2(\rho_1\rho_F/2) . \end{aligned} \tag{3.117}$$

□

Lemma 3.8: Suppose that Assumption 3.4 holds and that $\theta(k;\theta_0,k_0) \in \Theta$ for all $k \in [k_0,k_1+N-1]$. Then, for each $\epsilon \in (0,\epsilon_4)$ the averaged trajectory $\hat{\theta}(k;\theta_0,k_0)$ satisfies the ordinary difference equation

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \epsilon\overline{f}(\hat{\theta}(k)) + f_2(k;\theta_0,k_0,\epsilon) \tag{3.118}$$

for all $k \in [k_0,k_1-1]$ where f_2 satisfies

$$|f_2(k;\theta_0,k_0,\epsilon)| \leq \epsilon\,\kappa(\frac{1}{N}) + \epsilon(\epsilon N-\epsilon)\rho_1\rho_F + \epsilon^2\rho_\epsilon \tag{3.119}$$

for all $k \in [k_0,k_1-1]$.

Proof: From the definition of $\hat{\theta}$ it follows that for all $k \in [k_0,k_1-1]$

$$\hat{\theta}(k+1;\theta_0,k_0)-\hat{\theta}(k;\theta_0,k_0) = \frac{1}{N}(\theta(k+N;\theta_0,k_0)-\theta(k;\theta_0,k_0))$$
$$= \frac{1}{N}\sum_{i=0}^{N-1}\epsilon f(k+i,\theta(k+i;\theta_0,k_0);\epsilon)$$
$$= \frac{1}{N}\sum_{i=0}^{N-1}\epsilon f(k+i,\theta(k;\theta_0,k_0);0)$$
$$+ \frac{1}{N}\sum_{i=0}^{N-1}[\epsilon f(k+i,\theta(k+i;\theta_0,k_0);0)-\epsilon f(k+i;\theta(k;\theta_0,k_0);0)] \tag{3.120}$$
$$+ \frac{1}{N}\sum_{i=0}^{N-1}[\epsilon f(k+i,\theta(k+i;\theta_0,k_0);\epsilon)-\epsilon f(k+i,\theta(k+i;\theta_0,k_0);0)]$$
$$= \epsilon\bar{f}(\hat{\theta}(k;\theta_0,k_0)) + f_2(k;\theta_0,k_{,0},\epsilon)$$

where $f_2 = f_{21} + f_{22} + f_{23} + f_{24}$ with

$$f_{21}(k;\theta_0,k_0,\epsilon) = \left(\frac{1}{N}\sum_{i=0}^{N-1}\epsilon f(k+i,\hat{\theta}(k;\theta_0,k_0);0)\right) - \epsilon\bar{f}(\hat{\theta}(k;\theta_0,k_0)) \tag{3.121}$$

$$f_{22}(k;\theta_0,k_0,\epsilon) = \frac{1}{N}\sum_{i=0}^{N-1}\left(\epsilon f(k+i,\theta(k;\theta_0,k_0);0)-\epsilon f(k+i,\hat{\theta}(k;\theta_0,k_0);0)\right) \tag{3.122}$$

$$f_{23}(k;\theta_0,k_0,\epsilon) = \frac{1}{N}\sum_{i=0}^{N-1}\left(\epsilon f(k+i,\theta(k+i;\theta_0,k_0);0)-\epsilon f(k+i,\theta(k;\theta_0,k_0);0)\right) \tag{3.123}$$

$$f_{24}(k;\theta_0,k_0,\epsilon) = \frac{1}{N}\sum_{i=0}^{N-1}\left(\epsilon f(k+i,\theta(k+i;\theta_0,k_0);\epsilon)-\epsilon f(k+i,\theta(k+i;\theta_0,k_0);0)\right). \tag{3.124}$$

By Assumption 3.4 f_{21} is bounded by

$$|f_{21}(k;\theta_0,k_0,\epsilon)| \leqslant \epsilon\,\kappa(\frac{1}{N}). \tag{3.125}$$

Using Lemma 3.6 it follows that

$$|f_{22}(k;\theta_0,k_0,\epsilon)| \leqslant \frac{1}{N}\sum_{i=0}^{N-1}\epsilon\rho_1(\epsilon N-\epsilon)(\rho_F/2)$$
$$= \epsilon(\epsilon N-\epsilon)(\rho_1\rho_F/2). \tag{3.126}$$

Similarly to the proof of Lemma 3.6 we obtain the bound for f_{23}

$$|f_{23}(k;\theta_0,k_0,\epsilon)| \leqslant \frac{1}{N}\sum_{i=0}^{N-1}\epsilon\rho_1\epsilon\rho_F i = \epsilon(\epsilon N-\epsilon)(\rho_1\rho_F/2). \tag{3.127}$$

Finally we bound f_{24} by

$$|f_{24}(k;\theta_0,k_0,\epsilon)| \leq \frac{1}{N}\sum_{i=0}^{N-1} \epsilon^2\rho_\epsilon = \epsilon^2\rho_\epsilon . \tag{3.128}$$

The triangle inequality and (3.125)-(3.128) imply (3.119).

□

Thus, we have shown that $\overline{\theta}$ and $\hat{\theta}$ are both solutions of ordinary difference equations which are perturbations of

$$\theta(k+1) = \theta(k) + \epsilon\overline{f}(\theta) . \tag{3.129}$$

From (3.116) it is clear that the perturbation f_1 in (3.114) can be made arbitrarily small with respect to $\epsilon\overline{f}$ by taking ϵ sufficiently small. If we take $N = N(\epsilon) = \epsilon^{-r}$ for $r \in (0,1)$ it follows from (3.119) that the perturbation f_2 in (3.118) can also be made arbitrarily small with respect to $\epsilon\overline{f}$. Notice also from (3.110) that this choice of $N(\epsilon)$ allows us to make the distance between θ and $\hat{\theta}$ arbitrarily small. Hence, with these three lemmas we can generate many results relating solutions of the ordinary difference equation (3.104) and the ODE (3.108). We present first a result on finite time approximation.

Theorem 3.4: Suppose that Assumption 3.4 holds. Given any positive constant $\tau_1 < \infty$, no matter how large, and any $\sigma > 0$, no matter how small, if

$$\mathbf{B}(\sigma,\overline{\theta}(\tau;\theta_0)) \subseteq \Theta \tag{3.130}$$

for all $\tau \in [0,\tau_1]$, then there exists $\epsilon_5(\tau_1,\sigma) \in (0,\epsilon_4]$ such that for each $\epsilon \in (0,\epsilon_5)$ and any $k_0 \in \mathbf{Z}$

$$|\theta(k;\theta_0,k_0 - \overline{\theta}(t_k;\theta_0)| < \sigma \tag{3.131}$$

for all $k \in [k_0,k_0+k_1(\epsilon)]$ where $k_1(\epsilon) = \lfloor \tau_1/\epsilon \rfloor$ and $t_k = \epsilon(k-k_0)$.

Proof: Let $\epsilon_5(\tau_1,\sigma)$ be the smaller of ϵ_4 and the solution of

$$\sigma = \epsilon(\rho_F/2)(3 + e^{\rho_1\tau_1}) + (\kappa(\epsilon) + \epsilon\rho_1\rho_F + \epsilon\rho_\epsilon)\left(\frac{e^{\rho_1\tau_1}-1}{\rho_1}\right). \tag{3.132}$$

Let $N = N(\epsilon) = \lfloor \epsilon^{-} \rfloor$. We prove (3.131) by induction. Suppose that

$$|\theta(k;\theta_0,k_0 - \overline{\theta}(t_k;\theta_0)| < \sigma - \epsilon\rho_F \tag{3.131'}$$

holds for all $k \in [k_0, i]$ for some $i \in [k_0, k_0+k_1-1]$. Then $B(\epsilon^{1/2}\rho_F, \theta(i;\theta_0,k_0)) \subseteq \Theta$ which implies $\theta(k;\theta_0,k_0) \in \Theta$ for all $k \in [k_0, i+N]$. From this it follows that $\hat{\theta}(k;\theta_0,k_0)$ is well defined for all $k \in [k_0, i+1]$. Using Lemmas 3.6-3.8 we have for all $k \in [k_0, i]$

$$\begin{aligned}
|\hat{\theta}(k+1;\theta_0,k_0) - \overline{\theta}(t_{k+1};\theta_0)| &\leq (1+\epsilon\rho_1)\,|\hat{\theta}(k;\theta_0,k_0) - \overline{\theta}(t_k;\theta_0)| \\
&\quad + \epsilon(\pi(\epsilon^{1/2}) + \epsilon^{1/2}\rho_1\rho_F - \epsilon(\rho_1\rho_F/2) + \epsilon\rho_\epsilon) \\
&\leq (1+\epsilon\rho_1)^{k+1-k_0}\epsilon^{1/2}(\rho_F/2) \\
&\quad + (\pi(\epsilon^{1/2}) + \epsilon^{1/2}\rho_1\rho_F - \epsilon(\rho_1\rho_F/2) + \epsilon\rho_\epsilon)\left(\frac{(1+\epsilon\rho_1)^{k+1-k_0}-1}{\rho_1}\right) \qquad (3.133) \\
&< \sigma - \frac{3}{2}\epsilon^{1/2}\rho_F .
\end{aligned}$$

Lemma 3.6, the triangle inequality and (3.133) imply that (3.131′) holds for all $k \in [k_0, i+1]$. Hence, if (3.131′) holds for all $k \in [k_0, i]$ for any $i \in [k_0, k_0+k_1-1]$, then it holds for all $k \in [k_0, i+1]$. Since $\theta(k_0;\theta_0,k_0) = \theta_0 = \overline{\theta}(0;\theta_0)$ which guarantees that (3.131′) holds for $k=k_0$, (3.131′) holds for all $k \geq k_0$.

□

Remark 3.8: We can get a larger estimate for ϵ_5 by letting $N = \epsilon^{-r}$, defining $\epsilon_5'(\tau_1,\sigma,r)$ as the solution of

$$\sigma = \epsilon^{1-r}(\rho_F/2)(3 + e^{\rho_1\tau_1}) + (\kappa(\epsilon^{1-r}) + \epsilon^{1-r}\rho_1\rho_F + \epsilon\rho_\epsilon)\left(\frac{e^{\rho_1\tau_1}-1}{\rho_1}\right). \qquad (3.134)$$

and taking $\epsilon_5(\tau_1,\sigma) = \min\{\epsilon_4, \max_{r\in[0,1]}\{\epsilon_5'(\tau_1,\sigma,r)\}\}$. The proof using this estimate requires a change on the right-hand side of (3.131′) from $\sigma - \epsilon^{1/2}\rho_F$ to $\sigma - \epsilon^{1-r_*}\rho_F$ where $r_* \in [0,1]$ is the value for which ϵ_5' attains its maximum. In general, the bounds provided by Lemma 3.6-3.8 are so conservative that this procedure still results in a very conservative bound for ϵ_5.

□

Remark 3.9: If $f(k,\theta;0)$ is N-periodic, then we replace (3.132) by

$$\sigma = \epsilon N(\rho_F/2)(3 + e^{\rho_1\tau_1}) + (\epsilon N\rho_1\rho_F + \epsilon\rho_\epsilon)\left(\frac{e^{\rho_1\tau_1}-1}{\rho_1}\right). \tag{3.135}$$

□

In order to give a more complete connection with the ODE literature in the stochastic setting, we consider the system

$$\theta(k+1) = \theta(k) + \alpha_k f(k,\theta(k);\alpha_k), \qquad k \geqslant 0, \tag{3.136}$$

where α_k is a monotonically decreasing sequence which satisfies

$$\lim_{k\to\infty} \alpha_k = 0, \quad \alpha_k > \alpha_{k+i} \geqslant \alpha_k - \alpha_k^2 i, \quad \forall\, i \geqslant k \geqslant 0. \tag{3.137}$$

We note that $\alpha_k = (k+1)^{-r}$ for $r \in (0,1]$ satisfies (3.137). Letting $\theta'(k;\theta_0,k_0)$ denote the solution of (3.136) with initial data $\theta'(k_0) = \theta_0$, we have the following corollary to Theorem 3.4.

Corollary 3.4: Suppose that Assumption (3.4) holds and that (3.103) holds for all $\epsilon \in [0,\epsilon_4)$. Given positive constants τ_1 and σ, if (3.130) is satisfied for all $\tau \in [0,\tau_1]$, then there exists $M(\tau_1,\sigma)$ such that for any $k_0 \geqslant M$

$$|\theta'(k,\theta_0,k_0) - \overline{\theta}(t_k';\theta_0)| < \sigma \tag{3.138}$$

for all $k \in [k_0,k_0+k_1(\epsilon)]$ where $t_k' = \sum_{i=k_0}^{k-1} \alpha_i$ and $k_1(\epsilon)$ is the largest integer such that $t_{k_1}' \leqslant \tau_1$.

Proof: Clearly the bound (3.110) in Lemma 3.6 holds with θ replaced by θ' and ϵ replaced by α_{k_0}. Likewise, in the difference equation (3.114) in Lemma 3.7, we replace t_{k+1} by t_{k+1}', t_k by t_k', and ϵ by α_k. Hence, in the bound (3.116) we replace ϵ by α_{k_0}. In Lemma 3.8 we let $\hat{\theta}$ be the average of θ', redefine f_{22}, f_{23}, and f_{24}, add another term f_{25} to f_2, and replace $\epsilon\overline{f}(\hat{\theta}(k))$ by $\alpha_k\overline{f}(\hat{\theta}(k))$. In the term f_{21} we replace ϵ by $\alpha_k \leqslant \alpha_{k_0}$. Because we changed ϵ to α_k and not α_{k+i} in f_{21} we must add another term f_{25} to f_2.

$$f_{25}(k;\theta_0,k_0) = \frac{1}{N}\sum_{i=0}^{N-1} (\alpha_{k+i}-\alpha_k)\, f(k+i,\hat{\theta}(k;\theta_0,k_0);0). \tag{3.139}$$

Using (3.137) we bound f_{25} by

$$|f_{25}(k;\theta_0,k_0)| \leqslant \frac{1}{N}\sum_{i=0}^{N-1} \rho_F \alpha_{k_0}^2 i = \alpha_{k_0}(\alpha_{k_0}N - \alpha_{k_0})(\rho_F/2). \qquad (3.140)$$

Replacing the bound (3.119) with

$$|f_2(k;\theta_0,k_0)| \leqslant \alpha_{k_0}(\kappa(\frac{1}{N}) + (\alpha_{k_0}N-\alpha_{k_0})(\rho_1 + \frac{1}{2})\rho_F + \alpha_k\rho_\epsilon) \qquad (3.141)$$

the proof is completed as in the proof of Theorem 3.4 with $M(\tau_1,\sigma)$ chosen large enough so that $\alpha_M < \epsilon_4$ and

$$\sigma \geqslant \alpha_M^{1/2}(\rho_F/2)(3+e^{\rho_1\tau_1}) + (\kappa(\alpha_M^{1/2}) + \alpha_M^{1/2}(\rho_1 + \frac{1}{2})\rho_F + \alpha_M(\rho_\epsilon+\rho_F/2))\left(\frac{e^{\rho_1\tau_1}-1}{\rho_1}\right). \qquad (3.142)$$

Remark 3.10: As in Remark 3.8 we can get a smaller estimate of $M(\tau,\sigma)$ by letting $N = \alpha_M^{-r}$ and optimizing with respect to r. The estimate can also be improved if a specific α_k sequence is chosen or a decaying upper bound is used in (3.137).

□

Before giving results on infinite time approximation of $\theta(k)$ by $\overline{\theta}(t_k)$, we make several observations about the finite time results which we just presented. The conditions under which Theorem 3.4 and Corollary 3.4 are established are very mild, namely, that f is bounded and Lipschitzian in θ and ϵ and that the average $\overline{f}$ exists. We also emphasize that the same conditions are required for the constant gain case, (3.104), and the decaying gain case, (3.136), and that the same ODE, (3.108), is associated with the constant gain case and the decaying gain case. The difference between the two cases is that for the constant gain case we sample the ODE periodically at $t_k = \epsilon(k-k_0)$, whereas in the decaying gain case we sample the ODE at times $t_k' = \sum_{i=k_0}^{k-1} \alpha_i$ which are closer together as k increases. The price we pay for not using more information about f is that the approximation of $\theta(k;\theta_0,k_0)$ by $\overline{\theta}(t_k;\theta_0)$ and the approximation of $\theta'(k;\theta_0,k_0)$ by $\overline{\theta}'(t_k';\theta_0)$ are guaranteed only for a finite time interval. In fact, even if (3.130) holds for all $\tau \geqslant 0$, the approximations do not hold for all $\tau \geqslant 0$. For example, θ_0 could be an unstable equilibrium of (3.108) with the property that $\overline{\theta}(\tau;\theta_1)$ reaches the boundary of Θ in finite time for every $\theta_1 \neq \theta_0$.

If $f(k,\theta_0;\epsilon)$ is not identically equal to zero, then for some $k_1 > k_0$. $\theta(k;\theta_0,k_0) = \theta_1 \neq \theta_0$. Hence, for ϵ sufficiently small and $k \geqslant k_1$, $\theta(k;\theta_0,k_0) = \theta(k;\theta_1,k)$ follows $\overline{\theta}(\tau;\theta_1)$ to some neighborhood of the boundary of Θ in finite time which implies $\overline{\theta}(t_k;\theta_0) \equiv \theta_0$ is not a good approximation of $\theta(k;\theta_0,k_0)$ for all $k \geqslant k_0$.

Clearly we must make some additional assumptions about $\overline{f}(\theta)$ or the solutions of the ODE (3.108) in order to be able to relate $\overline{\theta}(t_k;\theta_0)$ and $\theta(k;\theta_0)$ over infinite intervals. We shall consider two different assumptions. For continuity with respect to the deterministic averaging theory, we prove an infinite time result under the assumption that the ODE has an asymptotically stable constant solution. For continuity with respect to the literature on the ODE method in adaptive systems and for ease of application in Chapter 4, we prove an infinite time result under the assumption that a Lyapunov function with certain properties exists.

We let θ_* in the interior of Θ be an asymptotically stable equilibrium of the ODE (3.108) and denote the region of attraction by Θ_A.

$$\Theta_A = \{\theta_0 \in \Theta : \overline{\theta}(\tau;\theta_0) \in \Theta \quad \forall \tau \geqslant 0 \quad \text{and} \ \lim_{\tau\to\infty} \overline{\theta}(\tau;\theta_0) = \theta_*\} . \tag{3.143}$$

As in Theorem 3.4 we need a ball around $\overline{\theta}$ to be in Θ; hence, we define a subset $\Theta_A'(\sigma)$ of Θ_A.

$$\Theta_A'(\sigma) = \{\theta_0 \in \Theta_A : B(\sigma,\overline{\theta}(\tau;\theta_0)) \subseteq \Theta \qquad \forall \tau \geqslant 0\} . \tag{3.144}$$

Theorem 3.5: Suppose that Assumption 3.4 holds and that θ_* in the interior of Θ is an asymptotically stable equilibrium of the ODE (3.108). Given any $\sigma > 0$ for which $\Theta_A'(\sigma)$ is not empty, there exists $\epsilon_6(\sigma) \in (0,\epsilon_4]$ such that for each $\epsilon \in (0,\epsilon_6)$, any $k_0 \in \mathbf{Z}$ and any $\theta_0 \in \Theta_A'(\sigma)$

$$|\theta(k;\theta_0,k_0) - \overline{\theta}(t_k;\theta_0)| < \sigma \tag{3.145}$$

for all $k \geqslant k_0$, where $t_k = \epsilon(k-k_0)$.

Proof: The proof follows Meerkov (1973). We use the following two consequences of the fact that θ_* is an asymptotically stable solution of the ODE (3.108). Let $\mu_0 > 0$ be chosen so that $B(\mu_0,\theta_*) \subseteq \Theta_A'(\sigma)$. There exists $\tau_*(\mu) < \infty$ such that for each $\mu \in (0,\mu_0]$ and for every $\theta_0 \in \Theta_A'(\sigma)$,

we have

$$|\overline{\theta}(\tau;\theta_0) - \theta_*| \leqslant \mu/2 \tag{3.146}$$

for all $\tau \geqslant \tau_*(\mu)$. There exists a continuous function $\delta(\mu)$ with $\delta(0) = 0$ such that for each $\mu \in (0,\mu_0]$ and for every θ_1 with $|\theta_1 - \theta_*| < \mu$

$$|\overline{\theta}(\tau;\theta_1) - \theta_*| < \delta(\mu) \tag{3.147}$$

for all $\tau \geqslant 0$. Letting $\epsilon_5'(\mu) = \epsilon_5(\tau_*(\mu) + \epsilon;\mu/2)$, we have for each $\mu \in \{(0,\mu_0) \cap (0,2\sigma)\}$ and any $k_0 \in \mathbb{Z}$

$$|\theta(k;\theta_0,k_0) - \overline{\theta}(t_k;\theta_0)| < \mu/2 \tag{3.148}$$

for all $k \in [k_0,k_1]$ where $k_1 = k_1(\mu,\epsilon) = \lfloor \tau_*(\mu)/\epsilon \rfloor + 1$ and $t_k = \epsilon(k-k_0)$. Hence, from the triangle inequality, (3.146), and (3.148) we have

$$|\theta(k;\theta_0,k_0) - \theta_*| < \mu \tag{3.149}$$

at $k = k_0+k_1$. Now we consider the interval $[k_0+k_1,k_0+2k_1]$. Let $\theta_1 = \theta(k_0+k_1;\theta_0,k_0)$. We note that $\theta(k;\theta_0,k_0) = \theta(k;\theta_1,k_0+k_1)$ for $k \geqslant k_0+k_1$ and that $|\theta_1-\theta_*| < \mu$. Applying Theorem 3.4 again, we get

$$|\theta(k;\theta_0,k_0) - \overline{\theta}(t_k-t_{k_1};\theta_1)| < \mu/2 \tag{3.150}$$

for all $k \in [k_0+k_1,k_0+2k_1]$. We point out that in (3.150) we are comparing $\theta(k;\theta_0,k_0)$ with a different trajectory of the ODE than in (3.148). Applying the triangle inequality, (3.147), and (3.150), we have

$$|\theta(k;\theta_0,k_0) - \theta_*| < \mu/2 + \delta(\mu) \tag{3.151}$$

for all $k \in [k_0+k_1,k_0+2k_1]$. Furthermore, from the definition of τ_* and k_1 we have

$$|\overline{\theta}(t_k-t_{k_1};\theta_1) - \theta_*| < \mu/2 \tag{3.152}$$

at $k = k_0+2k_1$; hence, (3.149) holds at $k = k_0+2k_1$. We prove by induction that (3.151) holds for all $k \geqslant k_0+k_1$ and that (3.149) holds at $k = k_0+nk_1$ for all integers $n \geqslant 1$ which implies that

(3.151) holds for all $k \geqslant k_o+k_1$. Suppose that (3.149) holds at $k = k_0+nk_1$ for any integer $n \geqslant 1$. Letting $\theta_1 = \theta(k_0+nk;\theta_0,k_0)$ it follows that (3.150) holds for all $k \in [k_0+nk_1,k_0+(n+1)k_1]$ with $t_k - t_{k_1}$ replaced by $t_k - t_{nk_1}$. Then the triangle inequality, (3.150), and (3.147) imply that (3.151) holds for all $k \in [k_0+nk_1,k_0+(n+1)k_1+k_1]$. Clearly, (3.152) holds at $k = k_0+(n+1)k_1+k_1$ with $t_k - t_{k_1}$ replaced by $t_k - t_{nk_1}$; hence, (3.149) holds at $k = k_0+(n+1)+k_1$. Since (3.149) holds at $k = k_0+k_1$, it follows that (3.149) holds at $k = k_0+nk_1$ for all $n \geqslant 1$ and (3.151) holds for all $k \geqslant k_0+k_1$. One more application of the triangle inequality gives

$$|\theta(k;\theta_0,k_0)-\overline{\theta}(t_k;\theta_0)| \leqslant |\theta(k;\theta_0,k_0)-\theta_*| + |\overline{\theta}(t_m;\theta_0)-\theta_*|\mu+\delta(\mu) \tag{3.153}$$

for all $k \geqslant k_0+k_1$. Then choosing $\mu_1(\sigma) \in (0,\mu_0]$ such that $\mu_1+\delta(\mu_1) \leqslant \sigma$ and taking $\epsilon_6(\sigma)=\epsilon_5(\mu_1)$ complete the proof.

□

Corollary 3.5: Suppose that Assumption 3.4 holds, that (3.103) holds for all $\epsilon \in [0,\epsilon_4)$, and that θ_* in the interior of Θ is an asymptotically stable equilibrium of the ODE (3.108). Given any $\sigma > o$ for which $\Theta_A'(\sigma)$ is not empty, there exists $M_1(\sigma) < \infty$ such that for any $k_0 \geqslant M$ and any $\theta_0 \in \Theta_A'(\sigma)$

$$|\theta'(k;\theta_0,k_0) - \overline{\theta}(t_k';\theta_0)| < \sigma - \alpha_{k_0}^{1/2}\rho_F \tag{3.154}$$

for all $k \geqslant k_0$, where $t_k' = \sum_{i=k_0}^{k-1} \alpha_i$. Furthermore,

$$\lim_{k\to\infty} \theta'(k;\theta_0 \; \;) = \theta_* \tag{3.155}$$

Proof: In order to establish (3.154) we follow the proof of Theorem 3.5, except that the repeated applications of Theorem 3.4 over intervals $[k_0+nk_1,k_0+(n+1)k_1]$ are replaced by repeated applications of Corollary 3.5 over intervals $[k_0+k_j',k_0+k_{j+1}']$ where k_1' is chosen so that $t_{k_1}' = \sum_{i=k}^{k_1'-1} \alpha_i \in [\tau_*(\mu),\tau_*(\mu)+\alpha_{k_n}]$ and k_{n+1}' is chosen so that $t_{k_{n+1}}' - t_{k_n}' = \sum_{i=k_n'}^{k_{n+1}'-1} \alpha_i \in [\tau_*(\mu),\tau_*(\mu)+\alpha_{k_{n+1}}]$ for $n \geqslant 1$. We note that (3.137) implies that $\lim_{k\to\infty} \sum_{i=0}^{k} \alpha_i = \infty$ which, in turn, implies that $k_n' < \infty$ for

each integer $n \in [0,\infty)$. To see that (3.155) is true we point out that in the proof of (3.154) we have shown that for any $k_0 \geq M_1(\sigma)$ and any $\theta_0 \in \Theta_A'(\sigma)$, $|\theta'(k;\theta_0,k_0)-\theta_*| < \sigma$ for all $k \geq k_1'$. Given any $\sigma' \in (0,\sigma)$ it follows that $\theta'(k;\theta_0,k_0) \in \Theta_A'(\sigma')$ for all $k \geq k_1'$. Choosing i_0 so that $i_0 \geq M_1(\sigma')$ and $i_0 \geq k_1'$ and taking $\theta_1 = \theta'(i_0,k_0,\theta_0)$ we have $|\theta'(i;\theta_0,k_0)-\theta_*| = |\theta'(i;\theta_1,i_0)-\theta_*| < \sigma'$ for all $i \geq i_1'$.

□

Our proofs of Theorem 3.5 and Corollary 3.5 use only the definition of uniform asymptotic stability. (Asymptotic stability of θ_* as a solution of the ODE (3.108) is uniform by virtue of the fact that (3.108) is time-invariant.) In an application of these results any demonstration of the asymptotic stability of θ_* is sufficient. Two of the most commonly used methods are verification via simulation or an application of Lasalle's theorem. Another approach to obtaining information about $\theta(k;\theta_0,k_0)$ that is valid for all $k \geq k_0$ is to first find a Lyapunov function which proves the asymptotic or exponential stability of a solution or an invariant set of he ODE (3.108) and, second, make use of this Lyapunov function in a study of (3.104) or (3.136). This approach does not explicitly relate θ or θ' to $\overline{\theta}$, but instead, provides information about the behavior of θ or θ' relative to the asymptotically stable solution or set. In the case of an asymptotically stable equilibrium arguments similar to the proofs of Theorem 3.5 or Corollary 3.5 can be applied to obtain the infinite time approximation results such as (3.145) or (3.154)-(3.155). In the case of an invariant set, the best one can hope for is to establish the existence of an invariant set for (3.104) or (3.136) and to apply Theorem 3.4 or Corollary 3.4 over finite time intervals. For simplicity and because it fits an application in Chapter 4, we illustrate this approach for the constant gain case under the following assumption.

Assumption 3.5: There exist scalars $c_0 > 0$, $\gamma_0 > 0$, and $\gamma_1 \in (0, c_0\gamma_0)$ and a vector $\theta_* \in \Theta$ such that

$$B(c_0,\theta_*) \subseteq \Theta \tag{3.156}$$

and

$$(\theta-\theta_*)^T\overline{f}(\theta)+\overline{f}^T(\theta)(\theta-\theta_*) \leqslant 2[-\gamma_0|\theta-\theta_*|^2+\gamma_1|\theta-\theta_*|] \tag{3.157}$$

for all $\theta \in \mathbf{B}(c_0,\theta_*)$.

□

Theorem 3.6: Suppose that Assumptions 3.4 and 3.5 hold. Given any $\sigma \in (0,c_0-\frac{\gamma_1}{\gamma_0})$, there exists $\epsilon_7(\sigma)\in(0,\epsilon_4]$ such that for each $\epsilon\in(0,\epsilon_7)$, any $k_0 \in \mathbf{Z}$ and any $\theta_0 \in \mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$

$$|\theta(k;\theta_0,k_0)-\theta_*| < (1-\epsilon\gamma_0)^{k-k_0}(|\theta_0-\theta_*|+\epsilon^{1/2}\rho_F)+[1-(1-\epsilon\gamma_0)^{k-k_0}](\frac{\gamma_1}{\gamma_0}+\sigma) \tag{3.158}$$

for all $k \geqslant k_0$.

Proof: Choose ϵ_7 so that

$$0<\epsilon_7\leqslant\epsilon_4\ ,\ \ \frac{\gamma_1}{\gamma_0}+\sigma\leqslant c_0-\epsilon_7^{1/2}\rho_F\ ,\ \ \frac{1}{\gamma_0}(\kappa(\epsilon_7^{1/2})+\epsilon_7^{1/2}\rho_1\rho_F+\epsilon_7\rho_\epsilon)+\epsilon_7^{1/2}(\rho_F/2)\leqslant\sigma\ . \tag{3.159}$$

Let $N = \epsilon^{-1/2}$. Suppose that $\theta(k;\sigma_0,k_0)\in\mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$ for all $k\in[k_0,i]$. Then $\theta(k,\theta_0,k_0)\in\Theta$ for all $k\in[k_0,i+N]$ which implies that $\hat{\theta}(k;\theta_0,k_0)$ is defined for $k\in[k_0,i+1]$. Using (3.157) and Lemma (3.8) we have for all $k\in[k_0,i]$

$$|\hat{\theta}(k+1;\theta_0,k_0)-\theta_*| \leqslant (1-\epsilon\gamma_0)|\hat{\theta}(k;\theta_0,k_0)-\theta_*| + |f_2(k;\theta_0,k_0,\epsilon)| \tag{3.160}$$

from which it follows that (3.158) holds for all $k\in[k_0,i+1]$. But this implies that $\theta(k;\theta_0,k_0)\in\mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$ for all $k\in[k_0,i+1]$. Thus, if $\theta(k;\theta_0,k_0)\in\mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$ for all $k\in[k_0,i]$ and an arbitrary i, then $\theta(k;\theta_0,k_0)\in\mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$ for all $k\in[k_0,i+1]$. Since $\theta_0\in\mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$, it follows that $\theta(k;\theta_0,k_0)\in\mathbf{B}(c_0-\epsilon^{1/2}\rho_F,\theta_*)$ for all $k\geqslant k_0$. Hence (3.160) and (3.158) hold for all $k\geqslant k_0$.

□

It is clear that this result can be combined with Theorem 3.3 to obtain results similar to Corollaries 3.2 and 3.3 for the system (3.1)-(3.2). It is also obvious that we could postulate many different assumptions about the behavior of $\overline{\theta}(\tau;\theta_0)$ or about $\overline{f}(\theta)$, and then, using Lemmas 3.6-3.8 derive results that apply to $\theta(k;\theta_0,k_0)$ or $\theta'(k;\theta_0,k_0)$. However, we feel that Theorems 3.4-3.6 and Corollaries 3.4-3.5 provide sufficient illustration of the use of Lemmas 3.6-3.8 for the reader to be

able to state and prove results which are applicable in each different situation.

We now consider the problem of establishing a connection between solutions of the ODE (3.108) and the system (3.104) or (3.136) when the input to the system is a sample path of a stochastic process. Since every sample path of the input process is a deterministic time sequence, we can check whether Assumptions 3.1-3.4 are satisfied on a sample path by sample path basis. For each sample path for which the assumptions are satisfied Theorems 3.4 and 3.5 hold. Hence, if we place conditions on the stochastic process which generates the input such that Assumptions 3.1-3.4 are satisfied for almost every sample path, then Theorems 3.4 and 3.5 hold with probability one (w.p.1). If, in addition, Assumption 3.5 is satisfied w.p.1, then Theorem 3.6 holds w.p.1.

Recall from the previous sections that the only property of the input that is used in Assumptions 3.1-3.3 is uniform boundedness. Therefore, we shall require almost every sample path to be a uniformly bounded sequence. The supremum over $k \in \mathbf{Z}$ can depend on the sample path. However, there should exist a single bound which holds for almost every sample path. It is easier to give sufficient conditions for Assumption 3.4 to be satisfied along any particular sample path than to say what conditions are necessary for Assumption 3.4 to be satisfied. Each sample path could, for example, be the sum of a finite number of sinusoids with different sample paths having different magnitudes, phases, frequencies, or numbers of sinusoids. The lack of dependence of $\overline{f}$ on k is most easily met by restricting the input process to be a stationary stochastic process. This is, in fact, a very natural restriction given that our goal is to reduce the study of (3.104) to the study of a time-invariant system. Since $\overline{f}$ is defined as a time average after a sample path has been chosen it can depend on the sample path. If $\overline{f}$ does depend on the sample path, then we have gained little or nothing by considering the input to be a sample path of a stochastic process. This claim follows from the fact that we then must study the ODE for each possible $\overline{f}$ in order to have a complete analysis, that is, we must make a series of studies for different deterministic inputs. The easiest way to avoid this complication is to restrict the input process to be ergodic. In this case the time average $\overline{f}(\theta)$ of $f(k,\theta;0)$ is equal to the ensemble average of $f(k,\theta;0)$, that is, the expected value

of $f(k,\theta;0)$. We summarize this discussion with the following lemma.

Lemma 3.9: Let the input to the system (3.1)-(3.2), that is, $w(k)$ and the k dependence of $f(k$ be a sample path of a stationary ergodic stochastic process. If Assumptions 3.1-3.3 hold for almost every sample path, then Assumption 3.4 holds with probability 1 and $\overline{f}(\theta) = E[f(k,\theta;0)]$. □

Remark 3.11: An interesting special case is when $f(k,\theta;\epsilon)$ is linear in θ, $f(k,\theta;\epsilon) = 0$, and Assumptions 3.4 and 3.5 hold with probability one for $\theta^* = 0$, $\gamma_1 = 0$. Theorem 3.6 then guarantees exponential convergence to an arbitrary small ball around the origin with probability one. This is related to the results of Bitmead and Anderson (1980a,b) and Shi and Kozin (1986). □

In many adaptive systems the function $f(k,\theta,x)$ in the parameter update (3.2) has the form

$$\begin{aligned} f(k,\theta,x) = {} & f_0(\theta)+f_1(\theta)\mathrm{col}(w(k)w^T(k))+f_2(\theta)\mathrm{col}(xx^T) \\ & + f_3(\theta)\mathrm{col}(xw^T(k))+f_4(\theta)w(k)+f_5(\theta)x \quad . \end{aligned} \tag{3.161}$$

With this form it follows that under the conditions of Lemma 3.9

$$\begin{aligned} \overline{f}(\theta) = {} & f_0(\theta)+f_1(\theta)\mathrm{col}(R_w(0))+f_2(\theta)\mathrm{col}(R_\nu(0,\theta) \\ & + f_3(\theta)\mathrm{col}(R_{\nu w}(0,\theta))+f_4(\theta)E[w(k)]+f_5(\theta)E[\nu(k,\theta)] \ . \end{aligned} \tag{3.162}$$

where R_w, R_ν, and $R_{\nu w}$ are the autocorrelation of w, the autocorrelation of ν and the crosscorrelation of ν and w, respectively. Because $\nu(k,\theta)$ is the output of a linear time-invariant system with stationary input $w(k)$, we can compute $R_\nu(0,\theta)$ and $R_{\nu w}(0,\theta)$ via Parseval's theorem using the power spectral density of w and the transfer function from w to ν. Hence, we can use the Theorems 3.3-3.6 and Lemma 3.9 to analyze the effect of the frequency content of $w(k)$ relative to the transfer function of the system (3.1) on the behavior of the system (3.1)-(3.2). This is discussed in more detail in Chapter 4.

3.6. Concluding Remarks

Following continuous-time proofs, we have established conditions for the existence of an exponentially attractive integral manifold for slow adaptation in discrete time. We have also given proofs of averaging theorems for the analysis of the on-manifold behavior of slowly adapting systems with deterministic inputs. Finally, we have discussed the relationship between the deterministic averaging results and the ODE method for the analysis of stochastic adaptive systems.

CHAPTER 4

REDUCED-ORDER MODEL REFERENCE ADAPTIVE CONTROL

4.1. Introduction

In Chapter 2 we established the existence of an integral manifold for a standard model reference adaptive control system, namely, the Narendra, Valavani (1978) controller for relative degree one. Then using the method of averaging we analyzed the behavior of the adaptive system when this controller is applied to a plant that does not satisfy the exact matching and SPR assumptions under which the controller was designed. For slow adaptation, we showed that the exact matching and SPR assumptions can be replaced by approximate matching, that is, small RMS error and "signal dependent SPR" assumptions.

This result, by itself, gives us the ability to design reduced-order model reference adaptive control systems because we can design the usual full-order controller for an assumed plant of lower dimension then the actual plant. However, such an approach suffers from the inadequacies of the usual full-order controller design. First, the number of adjustable parameters is determined by the assumed order of the plant and not by the number of adjustable parameters which the controller needs to achieve acceptable performance. Second, the usual procedures assume only that the plant is a black box of known order, hence, do not take advantage of much information which is usually available about the plant. Clearly, the two problems are related. By assuming that so little information is available about the plant and by making exact matching the only acceptable performance, the design is forced to include as many adjustable parameters as required by the assumed order of the plant.

In this chapter we present an alternative parameterization of the adjustable controller which separates the dynamic order of the controller and plant from the number of adjustable parameters. This provides the freedom to design a model reference adaptive controller with many fewer adjustable parameters than in the conventional design. The analysis then proceeds under assumptions which, in general, can be verified only in the analysis, simulation, and testing phases

of a control system design. The use of these assumptions provides our method with a very natural way to make use of information which is available prior to the commissioning of a control system. The analysis is carried out in several parts. We first establish the existence of an attractive integral manifold. Then, sufficient conditions for stability are derived using the averaging theorems of Chapter 3. We conclude this chapter with frequency domain interpretations of the stability conditions.

4.2. A Reduced-order Controller Parametrization

Earlier adaptive control schemes adjusted as many parameters as required by the assumed order of the plant. This choice was motivated by the desire of perfect matching in the disturbance-free case. However, even if the plant order were exactly known, the adjustment of more than a few of the most important parameters creates difficulties, especially when the inputs are not persistently exciting. We introduce a controller parametrization which permits a reduced number of adjustable parameters. One adjustable gain is assigned to each element in the vectors of transfer functions F_1 and F_2 and to the input r as shown in Fig. 4.1. The state representation of this parametrization with adjustable parameter vector $\theta = [\beta_0, \beta^T, \alpha^T]^T$ is given by

$$\begin{aligned} x_0(k+1) &= \Lambda_0 x_0(k) + b_0 \theta^T \phi(k) \\ x_1(k+1) &= b_1 c_0 x_0(k) + \Lambda_1 x_1(k) + b_1 d_0 \theta^T \phi(k) \\ x_2(k+1) &= \Lambda_2 x_2(k) + b_2 c_p x_p(k) + b_2 n_o(k) \\ x_p(k+1) &= b_p c_0 x_0(k) + \Lambda_p x_p(k) + b_p d_0 \theta^T \phi(k) + b_p n_i(k) \end{aligned} \tag{4.1}$$

where x_0, x_1, x_2, and x_p are the states of F_0, F_1, F_2, and W, respectively, where the regressor vector $\phi = [r, \phi^{1\,T}, \phi^{2\,T}]^T$ is given by

$$\phi(k) = \begin{bmatrix} r(k) \\ -C_1 x_1(k) \\ -C_2 x_2(k) - d_2 y_o(k) \end{bmatrix}, \quad y_o(k) = C_p x_p(k) + n_o(k) . \tag{4.2}$$

and where r(k) is the reference input and $n_i(k)$, $n_o(k)$ are disturbances. We have included an input

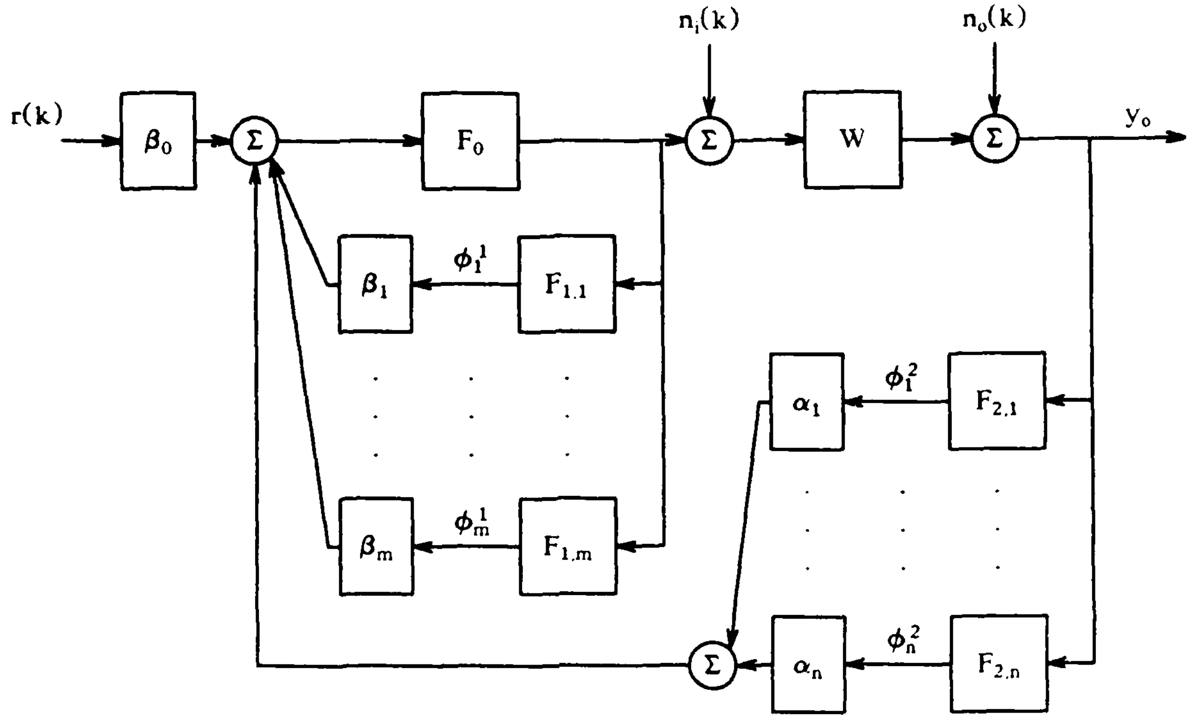

Fig. 4.1. Reduced-order parametrization with one gain per transfer function.

disturbance and an output disturbance with the idea that the input disturbance should represent inputs to the plant such as load variations and the output disturbance should represent, for example, measurement noise. The only output to which we refer is y_o representing the measured output. This must be taken into account when specifying the desired performance or evaluating the actual performance of the system. The compensator transfer functions F_0, F_1, and F_2 and the plant transfer function W are related to (4.1)-(4.2) by

$$F_0(z) = c_0(zI-\Lambda_0)^{-1}b_0 + d_0 , \quad F_1(z) = C_1(zI-\Lambda_1)^{-1}b_1$$
$$F_2(z) = C_2(zI-\Lambda_2)^{-1}b_2 + d_2 , \quad W(z) = c_p(zI-\Lambda_p)^{-1}b_p . \tag{4.3}$$

The number of adjustable parameters is determined by the number of compensator outputs, that is, $n_\alpha = m+n+1$ where C_1 has m rows and C_2 has n rows. The dynamic order of the compensators F_0, F_1, F_2 is at the designer's disposal and the number of parameters is not dependent on the order of the compensators. While we have combined all the states of F_1 and F_2, respectively, into x_1 and x_2

for a convenient state representation, we do not imply by (4.3) that the poles of each transfer function element of F_1 or F_2 are the same. That is, C_1, Λ_1, C_2, and Λ_2 can be block diagonal. However, we note that if

$$F_0(z) = 1\ , \quad F_1(z) = \frac{[1, z, \ldots, z^m]^T}{Z_m(z)}\ , \quad F_2(z) = \left[1\ , \frac{[1, z, \ldots, z^m]}{Z_m(z)}\right]^T . \tag{4.4}$$

where $Z_m(z)$ has order $m+1$, then this parametrization is equivalent to the full-order parametrization normally used in the design of adaptive controllers based on the black box assumption. Hence, our parametrization, which allows a reduced number of parameters, is more general, not less general, than the controller parametrizations usually encountered in the adaptive control literature.

Taking advantage of the freedom offered by this parametrization to work with a reduced number of parameters precludes, in general, the possibility of exact transfer function matching via the Bezout identity. We replace the goal of Bezout matching with the goal of minimizing the mean squared filtered tracking error between the reference model output $y_m(k)$ and the plant output $y_o(k)$ with the parameter held constant. We let the reference model transfer function and its output be

$$W_m(z) = c_m(zI - A_m)^{-1} b_m \ , \tag{4.5}$$

$$y_m(k) = \sum_{i=-\infty}^{k-1} c_m A_m^{k-1-i} b_m r(i)\ . \tag{4.6}$$

and we define the filtered tracking error $e(k)$ by

$$\begin{aligned} x_f(k+1) &= \Lambda_f x_f(k) + b_f(y_o(k) - y_m(k)) \\ e(k) &= c_f x_f(k)\ , \quad F(z) = c_f(zI - \Lambda_f)^{-1} b_f\ . \end{aligned} \tag{4.7}$$

Letting $e(k,\theta)$ denote $e(k)$ when the adjustable parameter is held constant at θ and the resulting linear time-invariant system (4.1),(4.7) is initialized with zero initial conditions at $k=-\infty$, that is, letting $e(k,\theta)$ be the steady-state response of $e(k)$ when the parameter is constant at θ, we make the

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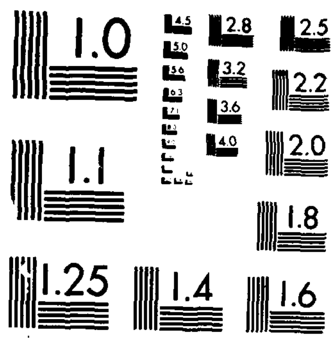

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following assumption.

Assumption 4.1: For the uniformly bounded signals $r(k)$, $n_i(k)$, $n_o(k)$ entering the system (4.1) and for the reference model $W_m(z)$, there exists θ^* which provides an isolated local minimum of the RMS tracking error

$$E(\theta) \triangleq \{\mathrm{avg}[e^2(\cdot,\theta)]\}^{1/2}$$
$$\triangleq \{\lim_{N\to\infty} \frac{1}{N} \sum_{i=k}^{k+N-1} e^2(i,\theta)\}^{1/2} \tag{4.8}$$

where the limit exists uniformly in $k \in \mathbb{Z}$.

□

Remark 4.1: Assumption 4.1 implies that the linear time-invariant system (4.1) with constant θ is exponentially stable at $\theta=\theta^*$ and in an open neighborhood around θ^*.

□

This assumption requires the use of *a priori* knowledge about the plant or range of possible plants $W(z)$. However, because the assumption is made for the linear time-invariant system (4.1) with constant θ, it requires essentially the same information that is necessary to design a fixed parameter controller with this structure. First the compensators F_0, F_1, and F_2 must be chosen so that each fixed plant in the range of possible plants can be stabilized for some value of θ. Then, taking into account the expected input signals, or designing the input signal, a reference model and an error filter are chosen which reflect an estimate of the achievable performance. That is, the reference model and error filter should be chosen so that the RMS error $E(\theta)$ can be made small. The advantages of a small $E(\theta^*)$ become clear in the sequel. While the satisfaction of these requirements may imply a significant off-line design effort, this effort is justified by the improved robustness in the on-line adaptation.

Remark 4.2: Small $E(\theta^*)$ imply transfer function matching. It only requires that the transfer functions be close at the dominant frequencies of the inputs to the system.

□

Although the number of parameters has been reduced, the structure of the proposed controller preserves the appearance of the parameter vector θ in (4.1) in the familiar parameter-regressor

product $\theta^T\phi$. Letting $y_o(k,\theta)$, $\phi(k,\theta)$ denote the steady-state of $y_o(k)$, $\phi(k)$, respectively, with the parameter vector held constant at θ, we note that the system (4.1) has the property

$$y_o(k,\theta) - y_o(k,\theta^*) = W_{CL}(\theta^*,z)[\phi^T(k,\theta)(\theta-\theta^*)] \tag{4.9}$$

with $\beta_o W_{CL}(\theta,z)$ being the transfer function from r to y_o

$$W_{CL}(\theta,z) = \frac{F_o(z)W(z)}{1 + F_o(z)(\beta^T F_1(z) + \alpha^T F_2(z)W(z))} , \tag{4.10}$$

and where, by the mixed k,z notation in (4.9) we mean that $y_o(k,\theta) - y_o(k,\theta^*)$ is the steady- state output of the transfer function $W_{CL}(\theta^*,z)$ with input $\phi^T(k,\theta)(\theta-\theta^*)$.

4.3. Parameter Update Law

We denote the system matrix of (4.1) by

$$\Lambda(\theta) = \bar{\Lambda} - b\theta^T C , \tag{4.11}$$

where the constant matrices $\bar{\Lambda}$, b, and C are

$$\bar{\Lambda} = \begin{bmatrix} \Lambda_0 & 0 & 0 & 0 \\ b_1c_0 & \Lambda_1 & 0 & 0 \\ 0 & 0 & \Lambda_2 & b_2c_p \\ b_pc_0 & 0 & 0 & \Lambda_p \end{bmatrix} , \quad b = \begin{bmatrix} b_0 \\ b_1d_0 \\ 0 \\ b_pd_0 \end{bmatrix} ,$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & C_2 & d_2c_p \end{bmatrix} . \tag{4.12}$$

From Remark 4.1 it follows that Assumption 4.1 implies Assumption 3.1 holds with $A(\theta)$ replaced by $\Lambda(\theta)$, that is, there exist a compact set Θ containing θ^* and an open neighborhood of θ^* and constants $\lambda_0 \in (0,1)$ and $K \geqslant 1$ such that

$$|\Lambda^i(\theta)| \leq K\lambda_0^i \quad \forall i \geq 0 \quad \forall \theta \in \Theta . \tag{4.13}$$

We assume that our *a priori* knowledge includes at least one point in the set Θ at which our parameter vector is initialized. Then, the task of parameter adaptation is to improve performance and track slow changes in the plant. Our use of slow adaptation has two advantages:

(i) the inherent stability of the fixed parameter controller quantified by (4.13) is preserved for slow variations of the plant parameters which otherwise could cause instability.

(ii) the parameters to not overreact to the misinformation that accompanies a nonzero minimum of the RMS tracking error.

The parameter θ is updated at every instant k by a small step which is proportional to the product of the filtered regressor vector ζ

$$\begin{aligned} \xi^i(k+1) &= A_{mf}\xi^i(k) + b_{mf}\phi_i(k) , \\ \zeta_i(k) &= c_{mf}\xi^i(k) , \quad i=1,2,\ldots, n_\theta , \\ F(z)W_m(z) &= c_{mf}(zI-A_{mf})^{-1}\, b_{mf} , \end{aligned} \tag{4.14}$$

the filtered tracking error e(k), and the step size ϵ

$$\theta(k+1) = \theta(k) - \epsilon\zeta(k)e(k) . \tag{4.15}$$

The choice of a constant filter to get ζ from ϕ and the choice of $F(z)W_m(z)$ as this constant filter have special significance for slow adaptation with the reduced parameterization (4.1). The motivation for this choice goes back to the method of sensitivity points, (Kokotovic, 1973). By this method the gradient of the output $y_0(k,\theta)$ with respect to the constant parameter θ is obtained by passing ϕ through the error filter F(z) and the exact closed-loop system transfer function $W_{CL}(\theta,z)$, Fig. 4.2. In particular, this holds at θ^* with ϕ^* and $W_{CL}(\theta^*,z)$. At θ^*, we obtain ζ by passing ϕ^* through $F(z)W_m(z)$. If the part of the filtered output $F(z)[y_0(k,\theta)]$ due to n_i and n_o is small relative to the part due to r, then by the definition of θ^*, $W_m(z)$ is near the best RMS approximation of $\beta_0^* W_{CL}(\theta^*,z)$; hence, ζ is proportional to a good approximation of the gradient. For this reason the filtered regressor ζ is also called the "pseudogradient" (Kokotovic, Medanic, Vuskovic, and Bingulac,

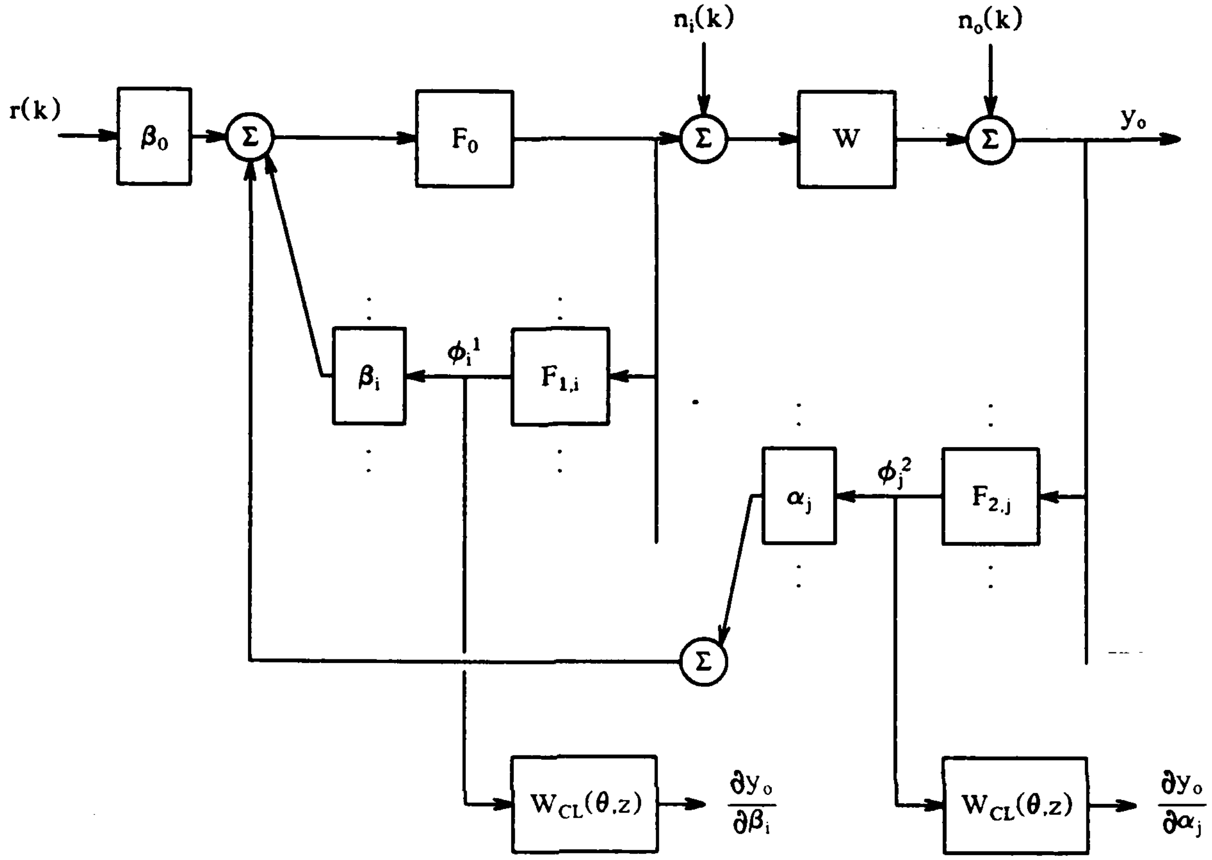

Fig. 4.2. Sensitivity points for obtaining the gradient of the output.

1966). This gradient approximation property will be used to show that the parameters converge to a neighborhood around θ^* with radius proportional to $E^2(\theta^*)$. Using the property (4.9) of the system (4.1) we see that the steady-state response of the $\zeta(k)e(k)$ with constant parameter θ, denoted by $\zeta(k,\theta)e(k,\theta)$, is given by

$$\zeta(k,\theta)e(k,\theta) = [F(z)W_m(z)\,\phi(k,\theta)]\,[F(z)W_{CL}(\theta^*,z)\phi(k,\theta)]^T\,(\theta-\theta^*) + e(k,\theta^*)\,. \qquad (4.16)$$

Notice that the first term on the right side of (4.16) is the product of a time-varying, θ-dependent matrix and the parameter error $\theta-\theta^*$. This structure is used to develop estimates of the region of attraction of an exponentially stable invariant set containing θ^*.

Letting $X^T = [x^T, x_f^T, \xi^T]$ where $x^T = [x_0^T, x_1^T, x_2^T, x_p^T]$ and $\xi^T = [\xi^{1\,T}, \xi^{2\,T}, \ldots, \xi^{n_\theta}]$ and letting $w(k) = [r(k), n_i(k), n_0(k), y_m(k)]$, we write the system (4.1), (4.7), (4.14), (4.15) in the form of (3.1)-(3.2),

$$X(k+1) = A(\theta(k))X(k) + B(\theta(k))w(k) \tag{4.17}$$

$$\theta(k+1) = \theta(k) + \epsilon\, f(X(k))\,, \tag{4.18}$$

where

$$A(\theta) = \begin{bmatrix} \Lambda(\theta) & 0 & 0 \\ A_{21} & \Lambda_f & 0 \\ A_{31} & 0 & A_{33} \end{bmatrix} \qquad B(\theta) = \begin{bmatrix} B_1(\theta) \\ B_2 \\ B_3 \end{bmatrix}$$

$$B_1(\theta) = \begin{bmatrix} \beta_\phi b_\phi & 0 & \alpha^T d_2 b_\phi & 0 \\ \beta_\phi b_1 d_\phi & 0 & \alpha^T d_2 b_1 d_\phi & 0 \\ 0 & 0 & b_2 & 0 \\ \beta_\phi b_b d_\phi & b_p & \alpha^T d_2 b_p d_\phi & 0 \end{bmatrix} \tag{4.19}$$

$$f(X) = -C_m X\, c_e X$$

with A_{21}, Λ_f, B_2, and c_e being the constant matrices corresponding to the tracking error filtering (4.7) and with A_{31}, A_{33}, B_3, and C_m being the constant matrices corresponding to the regressor vector filtering (4.14). The block triangular structure of $A(\theta)$ and (4.13) imply that Assumption 3.1 holds. Defining the frozen parameter response

$$\nu(k,\theta) \triangleq \sum_{i=-\infty}^{k-1} A^{k-1-i}(\theta)B(\theta)w(i) \tag{4.20}$$

it follows from the boundedness of w, the stability of $A(\theta)$, and the linear dependences of A and B on θ, that Assumption 3.2 holds. Finally, we note that $f(X)$ is quadratic in X; hence, Assumption 3.3 holds and Theorem 3.3 guarantees the existence of a local integral manifold of (4.17)-(4.18).

Theorem 4.1: Suppose that Assumption 4.1 holds. Then, the system (4.17)-(4.18) satisfies Assumptions 3.1-3.3; hence, for any given $D_0 > 0$, $\Delta_0 > 0$, $D_1 > D_0$, and $\lambda \in (\lambda_0, 1)$ there exists $\epsilon_4(D_0, \Delta_0, D_1, \lambda) > 0$ such that for each $\epsilon \in [0, \epsilon_4)$ there exists a function

$h(k,\theta;\epsilon) \in \mathbf{H}((\epsilon/\epsilon_4)D_0,(\epsilon/\epsilon_4)\Delta_0)$ with the following properties. Let $X(k),\theta(k)$ be the solution of (4.17)-(4.18) with initial data $X(k_0) = X_0$. $\theta(k_0) = \theta_0$ and let $g(k,\theta;\epsilon) = \nu(k,\theta) + h(k,\theta;\epsilon)$. If $\theta(k) \in \Theta$ for all $k \in [k_0,k_1]$, then

(i) $X_0 = g(k_0,\theta_0;\epsilon)$ implies $X(k) = g(k,\theta(k);\epsilon)$ $\forall k \in [k_0,k_1]$

(ii) $|X_0 - \nu(k_0,\theta_0)| \leqslant D_1/K$ implies that $\forall k \in [k_0,k_1]$

$$|X(k) - g(k,\theta(k);\epsilon| \leqslant K\lambda^{k-k_0} |X_0 - g(k_0,\theta_0;\epsilon)| . \tag{4.21}$$

□

Remark 4.3: If the vector of input signals $w(k)$ is N-periodic, then $g(k,\theta;\epsilon)$ is N-periodic in k.

□

Remark 4.4: In Theorem 4.1 we have not used in any essential way the fact that θ^* provides a minimum of $E(\theta)$ or that the limit in the definition of $E(\theta)$ exists uniformly with respect to k. We have used only the boundedness of $w(k)$ and the implied stability of $A(\theta)$.

□

4.4. Stability in the Slow Manifold: Averaging

The adaptive system (4.17)-(4.18) restricted to the slow manifold $M_\epsilon = \{k,\theta,X : X = g(k,\theta;\epsilon)\}$ evolves according to $X(k) = g(k,\theta(k);\epsilon)$ and

$$\theta(k+1) = \theta(k) + \epsilon f(g(k,\theta;\epsilon)) . \tag{4.22}$$

We apply the results of Section 3.5 to obtain sufficient conditions for (4.22) to possess an exponentially stable invariant set. We define the averaged system or ODE

$$\frac{d}{d\tau}\overline{\theta} = \overline{f}(\overline{\theta}) \tag{4.23}$$

where

$$\begin{aligned} \overline{f}(\theta) &\triangleq \mathrm{avg}[f(\nu(\cdot,\theta))] \\ &= -\mathrm{avg}[\zeta(\cdot,\theta)e(\cdot,\theta)] \end{aligned} \tag{4.24}$$

with $\phi(k,\theta)$, $\zeta(k,\theta)$, $e(k,\theta)$ denoting, respectively, the steady-state response of $\phi(k)$, $\zeta(k)$, $e(k)$,

that is,

$$\phi(k,\theta) = -\begin{bmatrix} r(k) \\ C\nu(k,\theta) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ d_2 n_0(k) \end{bmatrix} , \quad \zeta(k,\theta) = C_m\, \nu(k,\theta) \ , \quad e(k,\theta) = c_e\, \nu(k,\theta) \ . \tag{4.25}$$

In order to take advantage of the structure implied by (4.9), we introduce $v(k,\theta,\theta^*)$, the frozen parameter regressor vector $\phi(k,\theta)$ filtered by $F(z)W_{CL}(\theta^*,z)$,

$$v(k,\theta,\theta^*) = F(z)W_{CL}(\theta^*,z)\phi(k,\theta) \ . \tag{4.26}$$

Using (4.9) and (4.25) we write $f(g(k,\theta;0))$ as

$$f(\nu(k,\theta)) = -\zeta(k,\theta)v^T(k,\theta,\theta^*)(\theta-\theta^*) - \zeta(k,\theta)e(k,\theta^*) \ . \tag{4.27}$$

Defining

$$R(\theta,\theta^*) = \mathrm{avg}[\zeta(\cdot,\theta)v^T(\cdot,\theta,\theta^*)] \tag{4.28}$$

$$b(\theta,\theta^*) = \mathrm{avg}[\zeta(\cdot,\theta)e(\cdot,\theta^*)] \tag{4.29}$$

we rewrite $\overline{f}$ as

$$\overline{f}(\theta) = -R(\theta,\theta^*)(\theta-\theta^*) - b(\theta,\theta^*) \ . \tag{4.30}$$

From (4.30) it is clear that the averaged system (4.23) has the same structure as the averaged system which was studied in Section 2.6. Hence, we could develop the parallel results for the discrete-time adaptive system (4.17)-(4.18) using Theorems 3.4 and 3.5. However, we choose to obtain our sufficient conditions for stability with the more direct approach of Theorem 3.6. Letting

$$\gamma_1(K_1,\theta^*) = \max_{\theta\in B(K_1,\theta^*)} |||\,\zeta(\cdot,\theta)e(\cdot,\theta^*)\,||| \tag{4.31}$$

where $|||\cdot|||$ denotes the RMS value of the Euclidean norm,

$$|||\,\zeta(\cdot,\theta)\,||| \triangleq \{\mathrm{avg}[|\zeta(\cdot,\theta)|^2]\}^{1/2} \ . \tag{4.32}$$

we note that

$$|b(\theta,\theta^*)| \leq \gamma_1(K_1,\theta^*) \quad \forall\theta\in\mathbf{B}(K_1,\theta^*) \quad . \tag{4.33}$$

Theorem 4.2: Suppose that Assumption 4.1 holds, that $\mathbf{B}(K_1,\theta^*)\subseteq\Theta$, and that

$$R(\theta,\theta^*)+R^T(\theta,\theta^*) \geq 2\gamma_0 I>0 \quad \forall\theta\in\mathbf{B}(K_1,\theta^*) \quad . \tag{4.34}$$

Given any $D_0>0, \Delta_0>0, D_1>D_0$, and $\lambda\in(\lambda_0,1)$, if

$$\gamma_1(K_1,\theta^*) < K_1\gamma_0 \quad . \tag{4.35}$$

then for any $\sigma\in(0,K_1-\dfrac{\gamma_1}{\gamma_0})$ there exists $\epsilon_*(\sigma)\in(0,\epsilon_4(D_0,\Delta_0,D_1,\lambda)]$ such that for each $\epsilon\in(0,\epsilon_*)$, any $k_0\in\mathbf{Z}$, any $\theta_0\in\mathbf{B}(K_1-\epsilon^{1/2}\rho_F(D_0)-\epsilon\dfrac{D_1\rho_z(D_1)}{1-\lambda},\theta^*)$ and any $X_0\in\mathbf{B}(D_1/K,\nu(k_0,\theta_0))$ the solution $X(k)$, $\theta(k)$ of (4.17)-(4.18) with initial data $X(k_0)=X_0$, $\theta(k_0)=\theta_0$ satisfies (4.21) and

$$|\theta(k)-\theta^*| < (1-\epsilon\gamma_0)^{k-k_0}\left[\left|\theta_0-\theta^*|+\epsilon_{1/2}\rho_F+\epsilon\frac{D_1\rho_z(D_1)}{1-\lambda}\right|-\left|\frac{\gamma_1}{\gamma_0}+\sigma\right|\right]+(\frac{\gamma_1}{\gamma_0}+\sigma) \tag{4.36}$$

for all $k\geq k_0$.

Proof: In the proof of Lemma 3.6 we replace $f(k,\theta;\epsilon)$ by $f(X(k))$ and compute the new bound

$$\begin{aligned}|\hat{\theta}(k)-\theta(k)| &\leq \frac{1}{N}\sum_{i=0}^{N-1}|\theta(k+i)-\theta(k)| \leq \frac{1}{N}\sum_{i=0}^{N-1}\left|\sum_{j=0}^{i-1}[\epsilon\rho_F+\epsilon\rho_z(D_1)D_1\lambda^{k+j-k_0}]\right| \\ &\leq \frac{1}{N}\sum_{i=0}^{N-1}\left|\epsilon\rho_F i+\epsilon\frac{D_1\rho_z(D_1)}{1-\lambda}\right| \\ &\leq (\epsilon N-\epsilon)(\rho_F/2)+\epsilon\frac{D_1\rho_z(D_1)}{1-\lambda} \quad .\end{aligned} \tag{4.37}$$

In the proof of Lemma 3.8 we redefine f_{24} as

$$f_{24}(k;X_0,\theta_0,k_0,\epsilon) = \frac{1}{N}\sum_{i=0}^{N-1}\epsilon[f(X(k)-f(\nu(k,\theta(k)))] \quad . \tag{4.38}$$

and similarly to (4.37), we replace the bound (3.128) by

$$|f_{24}(k;X_0,\theta_0,k_0,\epsilon)| \leq \epsilon^2\rho_\epsilon + \frac{\epsilon}{N}\,\frac{D_1\rho_z(D_1)}{1-\lambda} \quad . \tag{4.39}$$

Noting that Lemma 3.6 was used to obtain (3.126) we replace it by

$$|f_{22}| \leq \epsilon(\epsilon N-\epsilon)(\rho_1\rho_F/2) + \epsilon^2 \frac{\rho_1 D_1 \rho_z(D_1)}{1-\lambda} \quad . \tag{4.40}$$

The proof is completed by following the proof of Theorem 3.6 with (3.160) replaced by

$$\begin{aligned} &\frac{\gamma_1}{\gamma_0}+\sigma \leq K_1-\epsilon^{1/2}\rho_F-\epsilon\frac{D_1\rho_z(D_1)}{1-\lambda} \\ &\frac{1}{\gamma_0}\left[\kappa(\epsilon^{1/2})+\epsilon^{1/2}\rho_1\rho_F+\epsilon\rho_\epsilon+(\epsilon^{1/2}+\epsilon\rho_1)\frac{D_1\rho_z(D_1)}{1-\lambda}\right]+\epsilon^{1/2}(\rho_F/2)+\epsilon\frac{D_1\rho_z(D_1)}{1-\lambda} \leq \sigma \quad . \end{aligned} \tag{4.41}$$

□

Corollary 4.1: Suppose that w(k) is a sample path of a stationary ergodic stochastic process. If the hypotheses of Theorem 4.2 are satisfied by almost every sample path of the process generating w(k), then the conclusions of Theorem 4.2 hold with probability 1.

□

Corollary 4.2: If w(k) is N periodic, then $\epsilon^{1/2}\rho_F(D_0)$ can be replaced with $\epsilon N\rho_F(D_0)$ in Theorem 4.2.

□

4.5. Frequency Domain Interpretation of Theorem 4.2

In this section we evaluate and interpret the stability condition (4.34). In the course of this study we relate the input signal frequency spectrum to the stability of the adaptive system (4.17)-(4.18) and investigate the effect of choosing reference models for which the minimum RMS error $E(\theta^*)$ is small.

For ease of exposition we assume temporarily that there are no disturbances, that is, $n_i(k)=n_o(k)\equiv 0$. In order to give an interpretation of the stability condition (4.34) in terms of the input signal spectrum and the pertinent transfer functions, we take r(k) as a finite sum of sinusoids,

$$r(k) = \sum_{\omega \in \Omega} r_\omega e^{j\omega k} \quad , \quad r_{-\omega} = \overline{r}_\omega \quad . \tag{4.42}$$

where $\overline{r_\omega}$ is the complex conjugate of r_ω, the set Ω has a finite number of elements, and $\omega \in \Omega$ implies $-\omega \in \Omega$. Denoting by $G(\theta,z)$ the vector of transfer functions from r to ϕ, that is,

$$G(\theta,z) = \begin{bmatrix} 1 \\ -C(zI-A(\theta))^{-1}b \end{bmatrix} , \tag{4.43}$$

we compute the Fourier series representation of ϕ, v, and ζ,

$$\phi(k,\theta) = \sum_{\omega \in \Omega} G(\theta,e^{j\omega})r_\omega e^{j\omega k} . \tag{4.44}$$

$$v(k,\theta,\theta^*) = \sum_{\omega \in \Omega} G(\theta,e^{j\omega})F(e^{j\omega})W_{CL}(\theta^*,e^{j\omega})r_\omega e^{j\omega k} . \tag{4.45}$$

$$\zeta(k,\theta) = \sum_{\omega \in \Omega} G(\theta,e^{j\omega})F(e^{j\omega})W_m(e^{j\omega})r_\omega e^{j\omega k} . \tag{4.46}$$

From (4.45) and (4.46) we calculate $R(\theta,\theta^*)$

$$R(\theta,\theta^*) = \sum_{\omega \in \Omega} G(\theta,e^{j\omega})G^T(\theta,e^{1j\omega}) |F(e^{j\omega})|^2 |r_\omega|^2 W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega}) . \tag{4.47}$$

and restate the stability condition (4.34) in frequency domain

$$\begin{aligned} 0 < 2\gamma_o I &\leqslant R(\theta,\theta^*) + R^T(\theta,\theta^*) \\ &= \sum_{\omega \in \Omega} G(\theta,e^{j\omega})G^T(\theta,e^{-j\omega}) |F(e^{j\omega})|^2 |r_\omega|^2 \mathrm{Re}(W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega})) , \quad \forall \theta \in B(K_1,\theta^*) . \end{aligned} \tag{4.48}$$

A necessary condition for (4.48) to be satisfied is that for some $\gamma > 0$

$$\begin{aligned} 0 < \gamma I &\leqslant \sum_{\omega \in \Omega} G(\theta,e^{j\omega})G^T(\theta,e^{-j\omega}) |r_\omega|^2 \\ &= \mathrm{avg}[\phi(\cdot,\theta)\phi^T(\cdot,\theta)] , \quad \forall \theta \in B(K_1,\theta^*) . \end{aligned} \tag{4.49}$$

which is clearly a persistent excitation (PE) condition on the regressor vector.

When the minimum RMS error $E(\theta^*)$ is small, we can show that a PE condition on the filtered regressor $\zeta(k,\theta)$ is a sufficient condition for the stability condition (4.48) to hold. Letting

$$w_1 = \max_{\theta \in B(K_1, \theta^*)} |||\zeta(\cdot,\theta)|||$$

$$g = \max_{\theta \in B(K_1, \theta^*)} \max_{\omega \in \Omega} |G(\theta, e^{j\omega})| \quad . \tag{4.50}$$

it follows that

$$|||\zeta(\cdot,\theta) - \beta_0^* v(\cdot,\theta,\theta^*)||| \leqslant gE(\theta^*) \ . \tag{4.51}$$

where we have the clear interpretation of g as the gain from r(k) to $\phi(k,\theta)$ and of $E(\theta^*)$ as a measure of transfer function mismatch $\beta_0^* F(z) W_{CL}(\theta^*, z) - F(z) W_m(z)$ at the frequencies $\omega \in \Omega$ of the reference input r(k). Using this bound and assuming a PE condition on the filtered regressor, that is,

$$\begin{aligned} 0 < \gamma I &\leqslant \text{avg}[\zeta(\cdot,\theta)\zeta^T(\cdot,\theta)] \\ &= \sum_{\omega \in \Omega} G(\theta, e^{j\omega})\, G^T(\theta, e^{-j\omega})\, |F(e^{j\omega})|^2 |W_m(e^{j\omega})|^2 |r_\omega|^2 \ . \quad \forall \theta \in B(K_1, \theta^*) \ . \end{aligned} \tag{4.52}$$

we have the following corollary to Theorem 4.2.

Corollary 4.3: Suppose that Assumption 4.1 holds, that $n_i = n_o \equiv 0$, that r(k) is given by (4.42), and that (4.52) holds. If $\beta_o^* > 0$ and

$$\gamma_o = \frac{1}{\beta_o^*}[\gamma - w_1 g E(\theta^*)] > 0 \quad . \tag{4.53}$$

then (4.34) is satisfied; hence, if (4.35) is satisfied, then the conclusions of Theorem 4.2 hold.

□

Thus, when the reference model and reference input are such that the error $E(\theta)$ can be made small, the stability condition (4.34) reduces to a PE condition (4.52) on the filtered regressor. An important point to remember is that this PE condition is checked pointwise in θ for constant values of $\theta \in B(K_1, \theta^*)$. That is, the vector $\zeta(k,\theta)$ which must be PE is the output of a linear time-invariant system driven by r(k); hence, the PE requirements on ζ are readily shifted to sufficient richness conditions on r. We see that r must contain frequencies for which G, F, and W_m are not too small and for which $\beta_o^* W_{CL}(\theta^*, j\omega) - W_m(j\omega)$ is small.

Equation (4.48) also points out one of the advantages of using the filtered regressor vector ζ in the update law rather than the regressor vector ϕ. Recall from Chapter 2 that the frequency domain interpretation of (4.34) was the "signal dependent SPR" condition on $W_{CL}(\theta^*, \cdot)$. From (4.48) notice that with regressor filtering by W_m we have a "signal dependent SPR" condition on $W_m(e^{j\omega})\, W_{CL}(\theta^*, e^{-j\omega})$. For small $E(\theta^*)$, $\beta_0^* W_{CL}(\theta^*, \cdot)$ is close to $W_m(z)$ at the frequencies $\omega \in \Omega$ and the positivity condition on $W_m(e^{j\omega}) W_{CL}(\theta^*, e^{-j\omega})$ is almost trivial. Hence, the regressor vector filtering removes the requirement that our model W_m be SPR.

The term $\gamma_1(K_1, \theta^*)$ defined in (4.33) is bounded by $\gamma_1 \leqslant w_1 E(\theta^*)$ which implies that the radius of the invariant set of the ODE (4.23) is $O(E(\theta^*))$. We use the fact that $\zeta(k, \theta^*)$ approximates the gradient $v(k, \theta^*, \theta^*)$ of $e(k, \theta)$ with respect to θ at $\theta = \theta^*$ in order to show that the invariant set actually has radius of $O(E^2(\theta^*))$. Let

$$w_2 = \max_{\theta \in \mathbf{B}(2w_1 E(\theta^*)/\gamma_0, \theta^*)} \left|\left|\left| \frac{\partial}{\partial \theta} \zeta(\cdot, \theta) \right|\right|\right| \quad . \tag{4.54}$$

Clearly, w_2 is bounded by a constant times v_1 (from Assumption 3.2). Letting

$$g^* = \max_{\omega \in \Omega} |G(\theta^*, e^{j\omega})| \quad . \tag{4.55}$$

we rewrite $b(\theta, \theta^*)$ as the sum of two terms

$$\begin{aligned} b(\theta, \theta^*) &= \operatorname{avg}[(\zeta(\cdot, \theta) - \zeta(\cdot, \theta^*)) e(\cdot, \theta^*)] \\ &\quad + \operatorname{avg}[(\zeta(\cdot, \theta^*) - \beta_0^* v(\cdot, \theta^*, \theta^*)) e(\cdot, \theta^*)] \end{aligned} \tag{4.56}$$

and bound $b(\theta, \theta^*)$ for all $\theta \in \mathbf{B}(2w_1 E(\theta^*)/\gamma_0, \theta^*)$

$$|b(\theta, \theta^*)| \leqslant \left(g^* + 2\frac{w_1 w_2}{\gamma_0} \right) E^2(\theta^*) \quad . \tag{4.57}$$

Using these bounds we show that after converging in finite time to the ball $\mathbf{B}(2w_1 E(\theta^*)/\gamma_0, \theta^*)$ the parameters converge exponentially to a smaller ball.

Corollary 4.4: Under the conditions of Theorem 4.2, if $n_1 = n_2 = 0$, $r(k)$ is given by (4.42), and

$$\left[g^* + 2\,\frac{w_1 w_2}{\gamma_0}\right] E(\theta^*) \leqslant \frac{w_1}{\gamma_0}\,, \quad \sigma < \frac{w_1}{\gamma_0^2}\, E(\theta^*) - \epsilon_{\rho_F}^{1/2} - \epsilon\, \frac{D_1 \rho_F(D_1)}{1-\lambda}\,. \tag{4.58}$$

then

$$|\theta(k) - \theta^*| \leqslant (1-\epsilon\gamma_0)^{k-k_1}\, 2\, \frac{w_1 E(\theta^*)}{\gamma_0} + [1-(1-\epsilon\gamma_0)^{k-k_1}] \left[\left[g^* + 2\,\frac{w_1 w_2}{\gamma_0} \right] \frac{E^2(\theta^*)}{\gamma_0} + \sigma \right] \tag{4.59}$$

for all $k \geqslant k_1(\epsilon)$ where

$$k_1(\epsilon) = \frac{1}{\epsilon\gamma_0} \left[\ln \left[K_1 - \frac{w_1 E(\theta^*)}{\gamma_0} - \sigma \right] - \ln \left[\frac{2 w_1 E(\theta^*)}{\gamma_0} - \epsilon_{\rho_F}^{1/2}(D) - \epsilon\, \frac{D_1 \rho_z(D_1)}{1-\lambda} \right] \right]. \tag{4.60}$$

□

In Chapter 2 without regressor filtering the best we could hope for was that the equilibrium of the ODE was $O(E(\theta^*))$ from θ^*. Hence, a second advantage of using regressor filtering over not using it is that it allows the slowly adapting system to converge to a smaller invariant set around the optimal parameter value.

The model reference adaptive system (4.17)-(4.18) also allows filtering of the measured error between y_0 and y_m by the transfer function $F(z)$. We drop the assumption of no disturbances in order to investigate the advantages of this error filtering. We suppose that the input disturbance n_i is a finite sum of sinusoids with frequencies in the set Ω_i,

$$n_i(k) = \sum_{\omega \in \Omega_i} n_{i\omega}\, e^{j\omega k}\,, \tag{4.61}$$

and that the output disturbance $n_o(k)$ is a finite sum of sinusoids with frequencies in the set Ω_o

$$n_o(k) = \sum_{\omega \in \Omega_o} n_{o\omega}\, e^{j\omega k}\,. \tag{4.62}$$

We allow Ω and Ω_i to have common elements but we assume that $\Omega \cap \Omega_o = \phi$ and $\Omega_i \cap \Omega_o = \phi$. Letting $G_i(\theta,z)$ and $G_o(\theta,z)$ be, respectively, the transfer functions from $n_i(k)$ to $\phi(k,\theta)$ and from $n_o(k)$ to $\phi(k,\theta)$, we compute $R(\theta,\theta^*)$

$$
\begin{aligned}
R(\theta,\theta^*) = &\sum_{\omega\in\Omega} G(\theta,e^{j\omega})\, G^T(\theta,e^{-j\omega})\ |F(e^{j\omega})|^2\ |r_\omega|^2\ W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega}) \\
&+ \sum_{\omega\in\Omega\cap\Omega_i} G(\theta,e^{j\omega})\, G_i^T(\theta,e^{-j\omega})\ |F(e^{j\omega})|^2 r_\omega\, \bar{n}_{i\omega}\ W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega}) \\
&+ \sum_{\omega\in\Omega\cap\Omega_i} G_i(\theta,e^{j\omega})\, G^T(\theta,e^{-j\omega})\ |F(e^{j\omega})|^2\ \bar{r}_\omega n_{i\omega}\ W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega}) \\
&+ \sum_{\omega\in\Omega_i} G_i(\theta,e^{j\omega})\, G_i^T(\theta,e^{-j\omega})\ |F(e^{j\omega})|^2\ |n_{i\omega}|^2\ W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega}) \\
&+ \sum_{\omega\in\Omega_o} G_o(\theta,e^{j\omega})\, G_o^T(\theta,e^{-j\omega})\ |F(e^{j\omega})|^2\ |n_{o\omega}|^2\ W_m(e^{j\omega})W_{CL}(\theta^*,e^{-j\omega})\,.
\end{aligned}
\tag{4.63}
$$

Letting $W_i(\theta,z)$ and $W_o(\theta,z)$ denote, respectively, the transfer function from $n_i(k)$ to $y_o(k)$ and from $n_o(k)$ to $y_o(k)$, we compute the RMS error $E(\theta)$

$$
\begin{aligned}
E(\theta) = &\sum_{\omega\in\Omega} |F(e^{j\omega})|^2\ |r_\omega|^2\ |\beta_o W_{CL}(\theta,e^{j\omega}) - W_m(e^{j\omega})|^2 \\
&+ \sum_{\omega\in\Omega_i\cap\Omega} |F(e^{j\omega})|^2\ r_\omega \bar{n}_{i\omega}\ W_i(\theta,e^{-j\omega})(\beta_0 W_{CL}(\theta,e^{j\omega}) - W_m(e^{j\omega})) \\
&+ \sum_{\omega\in\Omega_i\cap\Omega} |F(e^{j\omega})|^2\ \bar{r}_\omega n_{i\omega}\ W_i(\theta,e^{j\omega})(\beta_o W_{CL}(\theta,e^{-j\omega}) - W_m(e^{-j\omega})) \\
&+ \sum_{\omega\in\Omega_i} |F(e^{j\omega})|^2\ |W_i(\theta,e^{j\omega})|^2\ |n_{i\omega}|^2 \\
&+ \sum_{\omega\in\Omega_o} |F(e^{j\omega})|^2\ |W_o(\theta,e^{j\omega})|^2\ |n_{o\omega}|^2\,.
\end{aligned}
\tag{4.64}
$$

From (4.64) it is clear that minimizing $E(\theta)$ requires the controller to make $|W_i(\theta,e^{j\omega})|$ and $|W_o(\theta,e^{j\omega})|$ small at frequencies $\omega \in \Omega_i \cup \Omega_o$ while also making $|\beta_o W_{CL}(\theta,e^{j\omega}) - W_m(e^{j\omega})|$ small at frequencies $\omega \in \Omega$. If we further assume that the output disturbance contains only high frequencies and that the reference input and input disturbances contain only low frequencies, that is,

$$
0 \leqslant \max_{\omega\in\Omega_i\cup\Omega} |\omega| < \omega_c < \min_{\omega\in\Omega_o} |\omega| < \pi\,, \tag{4.65}
$$

and that $n_o(k)$ is measurement error which is to be ignored rather than compensated, then we can take advantage of $F(z)$ to make our cost functional $E(\theta)$ compatible with our objective and to improve robustness of the parameter update. We simply choose $F(z)$ so that $|F(e^{j\omega})|$ is zero or very small for all $\omega \in (\omega_c,\pi)$. This removes the effect of $n_o(k)$ from both $E(\theta)$ and $R(\theta,\theta^*)$. Notice that only the magnitude of $F(e^{j\omega})$ appears in $E(\theta)$ and $R(\theta,\theta^*)$. This implies that we have no

that only the magnitude of $F(e^{j\omega})$ appears in $E(\theta)$ and $R(\theta,\theta^*)$. This implies that we have no constraints on the phase characteristic of $F(e^{j\omega})$ when we design its magnitude characteristic.

Error filtering is, however, not a cure for all problems. If we want our controller to make the response of the plant due to the input disturbance $n_i(k)$ small, then $|F(e^{j\omega})|$ should be nonzero for all $\omega \in \Omega_i$ so that our cost functional reflects our objective. However, if $|F(e^{j\omega})|$ is not zero for each $\omega \in \Omega_i$, then we want to have $Re(W_m(e^{j\omega}) W_{CL}(\theta^*,e^{-j\omega})) > 0$ and $Im(W_m(e^{j\omega}) W_{CL}(\theta^*,e^{-j\omega}))$ small for all $\omega \in \Omega_i$ so that $R+R^T$ stays positive. This may be difficult to ensure or justify for frequencies $\omega \in \Omega_i$ which are not close to the frequencies in Ω. Hence, in the design of the adaptive system, $|F(e^{j\omega})|$ large to include $n_{i\omega}$ in the cost functional $E(\theta)$ may have to be traded off against $|F(e^{j\omega})|$ small to keep $R+R^T$ positive. Theorem 4.2 and the expressions (4.63) and (4.64) offer guidelines for this step in the design.

Remark 4.5: If the input $\omega(k)$ is generated by a stationary ergodic random process, then the frequency domain interpretation remains valid, but the sums over finite sets are replaced by integrals over $(-\pi,\pi)$ and the Fourier series coefficients are replaced by the spectral density. □

4.6. Concluding Remarks

We have presented an adaptive control scheme with a controller parameter parametrization that allows for the design of model reference adaptive control systems with a reduced number of parameters. Verification of the stability conditions for the parameter update may require significant off-line design effort or *a priori* knowledge. This, however, should be considered as an opportunity rather than a burden because it allows the designer to use available information to reduce the number of parameters in the adaptive control system.

CHAPTER 5

DESIGN OF SLOWLY ADAPTING CONTROL SYSTEMS: AN EXAMPLE

5.1. Introduction

Successful application of the model reference adaptive control system introduced in Chapter 4 consists of two separate developments, both of which relate to the control system (4.1) with constant values of the adjustable gains θ. First, the compensator blocks F_0, F_1, and F_2 must be designed so that the fixed gain control system can be tuned to give acceptable performance by adjusting only the gains θ. Since the main reason for having adaptive control is to combat parameter uncertainty or variability in the plant, the possibility to tune the control system by adjusting only the gains θ should exist for the entire range of possible plants. The second step is to design the reference model, the error filter, and ,if applicable, the input signals so that the θ^* which minimizes the RMS filtered tracking error $E(\theta)$ provides good tuning of the control system. The value of θ^* which minimizes $E(\theta)$ should depend on which plant in the range of possible plants is used, but the property that θ^* provides good tuning of the control system should hold for any plant in the range of possible plants. We remark that if θ^* is the same for each plant, then adaptive control is not necessary. Following Kokotovic, Medanic, Vuskovic, and Bingulac (1966), we shall say that a controller is *compatible* if it can be tuned for each possible plant by changing only θ.

These two steps can be generalized to provide guidelines for the design of slowly adapting control systems.

(1) Given the range of possible linear time-invariant plants, choose a controller parametrization with adjustable parameter vector θ such that, for each possible plant,

(a) the fixed parameter controller is compatible, and

(b) if the closed-loop system is written in the state space form

$$x(k+1) = A(\theta)x + B(\theta)w(k) \quad , \tag{5.1}$$

then $A(\theta)$ and $B(\theta)$ are differentiable with Lipschitzian derivatives.

(2) Given such a controller, find a cost functional $J(\theta)$ such that

(a) for each possible plant, the θ^* which minimizes $J(\theta)$ provides acceptable tuning of the fixed parameter control system, and

(b) $J(\theta)$ is differentiable and its derivative is Lipschitzian.

(3) Construct filters with state ξ, inputs x and w, and output ζ, and construct a parameter update law

$$\theta(k+1) = \theta(k) + \epsilon f(w(k),\theta(k),x(k),\zeta(k)) \tag{5.2}$$

so that, in the averaged system

$$\frac{d}{d\tau}\bar{\theta} = \bar{f}(\bar{\theta}) \quad , \tag{5.3}$$

where

$$\bar{f}(\theta) \triangleq \mathrm{avg}[\ f(w(\cdot),\theta,x(\cdot,\theta),\zeta(\cdot,\theta))\ ] \quad , \tag{5.4}$$

$\bar{f}$ satisfies

$$\bar{f}^T = -\frac{\partial J}{\partial \theta} \quad \text{or} \quad \bar{f}^T \cong -\frac{\partial J}{\partial \theta} \quad . \tag{5.5}$$

Assuming that acceptable performance implies all eigenvalues of $A(\theta^*)$ are strictly inside the unit circle, it follows that we can establish the existence of an exponentially attractive integral manifold in a ball around θ^* for the slowly adapting control system (5.1)-(5.2). In the manifold we apply averaging to investigate the evolution of the parameters. Assuming isolated local minima, (5.5) quarantees that solutions of the the ODE (5.3) beginning close enough to θ^* converge to θ^* or a small invariant set containing θ^*.

We point out that step (1) is required in the design of any linear time-invariant controller which is applied to an uncertain plant, to a nonlinear plant linearized at different operating points, or to different copies of the same product. Step (2) is related to the off-line tuning of such a controller. If the controller has more than three parameters, manual tuning is often a difficult task. Automated tuning requires the specification of a cost functional to be minimized. However, in off-line tuning with a human supervisor, good performance does not have to occur at the minimum of $J(\theta)$. The supervisor can monitor performance during each step of an iterative tuning procedure

and stop when the performance is good. Since the adaptive control system is not supposed to need a supervisor, the cost functional for adaptive control must be chosen with more care than the one for supervised automatic tuning. The ability to construct the required filters and the function f in step (3) is often related to the ability to do off-line automatic tuning because the cost functional $J(\theta)$ often has the form

$$J(\theta) = \text{avg}[\, J_1(x(\cdot,\theta))\,] \quad . \tag{5.6}$$

In this case, we take

$$f^T(w(k),\theta(k),x(k),\zeta(k)) = -\frac{\partial J_1}{\partial x}\,\frac{\partial x}{\partial \theta} \quad . \tag{5.7}$$

In Chapter 4, for example, we used $J_1 = .5(y(k)-y_m(k))^2$ so that $\frac{\partial J_1}{\partial y} = y(k)-y_m(k)$ and ζ was used to approximate $\frac{\partial y}{\partial \theta}$.

In this chapter we use these guidelines to design a slowly adapting control system for a simplified model of gasoline engine idle-speed control.

5.2. Problem Statement

The plant and controller parametrization are given; see Fig. 5.1. The plant uncertainty is parametrized by the vector of plant parameters $p = [p_1 \;\; p_2 \;\; p_3]^T$ with nominal value $[0.67 \;\; 0.017 \;\; 0.75]^T$. Each element of p can vary by 30% of its nominal value. The elements θ_1, θ_2, and θ_3 of the controller parameter vector θ are the proportional gain from the output y to the input u_2, the proportional gain from y to the input u_1, and the integral gain from y to the input u_1, respectively. A state space representation of the closed-loop system is

$$x(k+1) = \begin{bmatrix} .994 & p_2 & 0 & 0 & 0 & 0 \\ -10\theta_1-.5p_3 & 0 & .5 & .4 & .25 & 0 \\ -(1+p_1)p_3 & 0 & p_1 & .8(1+p_1) & .5(1+p_1) & 0 \\ 0 & 0 & 0 & 0 & .25 & 0 \\ -\theta_2 & 0 & 0 & -.28 & .4 & -\theta_3 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} d(k) \tag{5.8}$$
$$y(k) = [\,1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0\,]x(k) \quad .$$

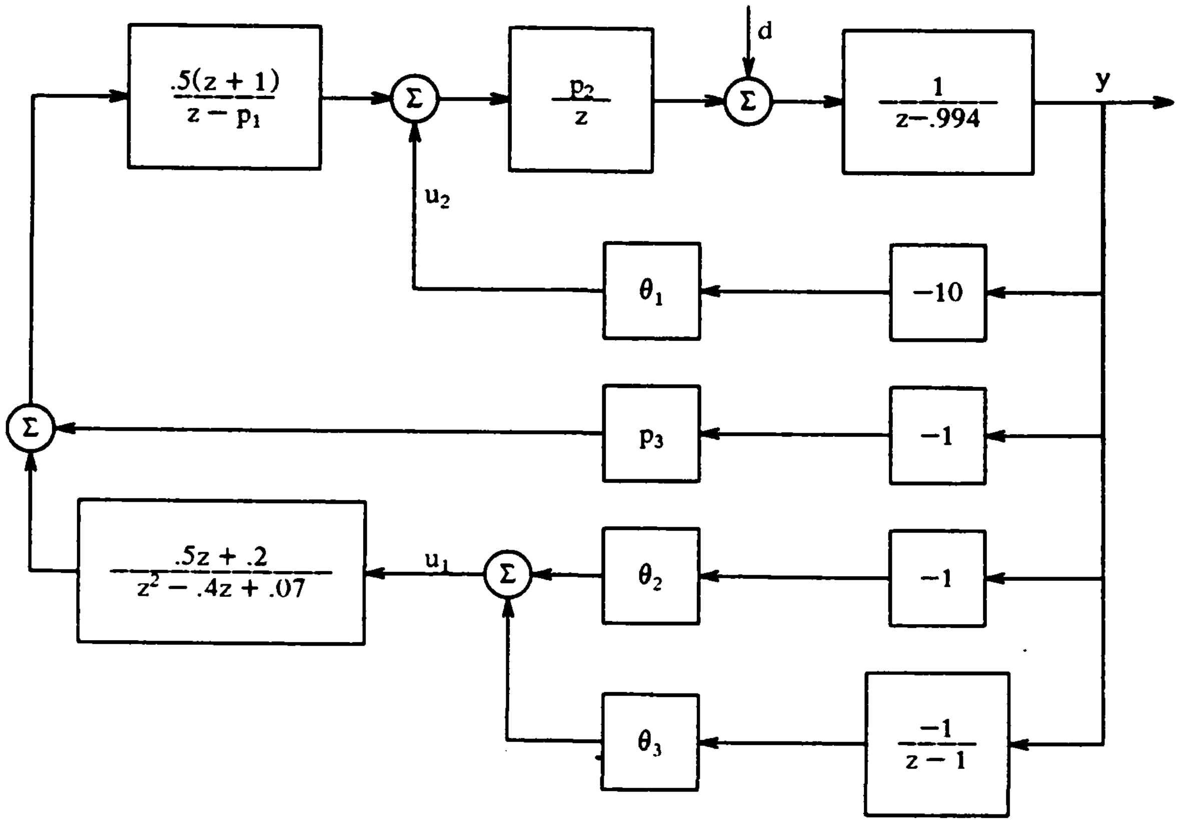

Fig. 5.1. Block diagram of the closed-loop system (5.8) with plant parameters p_1, p_2, and p_3 and controller parameters θ_1, θ_2, and θ_3.

The desired value of the output is zero. However, the system is subject to infrequent (separated by more than 50 samples) step changes in the unmeasured disturbance d, representing load changes. Good tuning of the controller should achieve several objectives simultaneously. The response of y to a unit step change in d at time k_0 should have magnitude less than 0.1 for all $k \geqslant k_0 + 25$ and magnitude less than 0.01 for all $k \geqslant k_0 + 50$. The response should be well damped. The closed-loop eigenvalues should all have magnitudes less than 0.9 so that integrator windup is not a problem. These three objectives were stated in order of increasing importance.

In terms of the guidelines presented in the introduction, the choice of the controller parametrization has been given and it satisfies the smoothness condition (1b). The compatibility requirement (1a) that the controller can be tuned for different values of the plant parameter p is to

be checked by actually tuning the controller for different values of p. The development of a cost functional with the properties described in (2) is presented as an iterative procedure in which an appropriate cost functional is determined for tuning the plant with the nominal value of p, and then, it is tested to verify that it provides good tuning for all possible values of p. The construction of a parameter update law is straightforward because the the cost functional has the form (5.6).

5.3. Tuning of the Nominal Plant

Using the method of sensitivity points (Kokotovic, 1973), the gradient of the output of y(k) with respect to constant controller parameters is $v^T(k) = [\ v_1(k) \quad v_2(k) \quad v_3(k)\ ]$, where v_1, v_2, and v_3 are signals at the indicated points in the block diagram of the sensitivity model, Fig. 5.2. A state

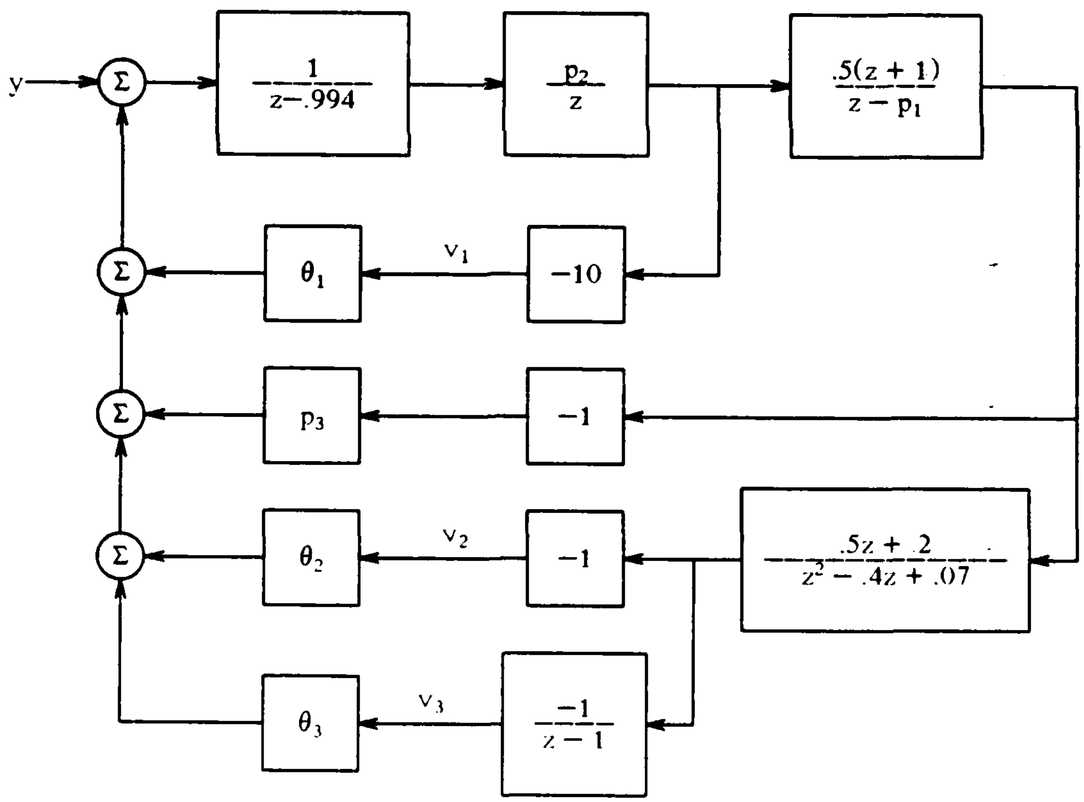

Fig. 5.2. Block diagram of the sensitivity model (5.9) for the system (5.8) showing the sensitivity points v_1, v_2, v_3.

space represention of the sensitivity model is

$$\xi(k+1) = \begin{bmatrix} .994 & -p_2(10\theta_1+.5p_3) & -.5p_3 & -.8\theta_2 & -.5\theta_2 & -\theta_3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1+p_1)p_2 & p_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .25 & 0 \\ 0 & .5p_2 & .5 & -.28 & .4 & 0 \\ 0 & 0 & 0 & .8 & .5 & 1 \end{bmatrix} \xi(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} y(k) \tag{5.9}$$

$$v(k) = \begin{bmatrix} 0 & -10p_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -.8 & -.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \xi(k) \ .$$

Because the plant parameter vector p is unknown, this sensitivity model cannot be realized on line. We use it for off-line simulation studies. For an implementable algorithm, we shall use (5.9) with the given nominal value of p and the constant value of θ which gives good tuning for the nominal plant.

Since the input d is not measured, the usual model reference approach of using the squared *tracking error for the* cost functional results in

$$J = \text{avg}[\ y^2(\cdot)\ ]. \tag{5.10}$$

For testing candidate cost functionals, we let d(k) be a square wave of period 100 taking values 1 and 0. The average of y^2 is minimized for the nominal plant by the controller parameter value $\theta^* = [\ 5.4 \quad 10.8 \quad 3.7\ ]^T$. As shown in Fig. 5.3 the response is oscillatory. As one of the objectives is to have a well-damped response, something needs to be added to the cost functional to penalize the oscillations.

We used the parameter update law

$$\theta(k+1) = \theta(k) - \epsilon v(k)y(k) \tag{5.11}$$

and slow adaptation to search for θ^*. Examining the response for different values of θ along the trajectory of the slowly adapting system (5.8),(5.9),(5.11), we observe that for some values of the parameters the response is close to that of a well-damped second order system with two zeros. If

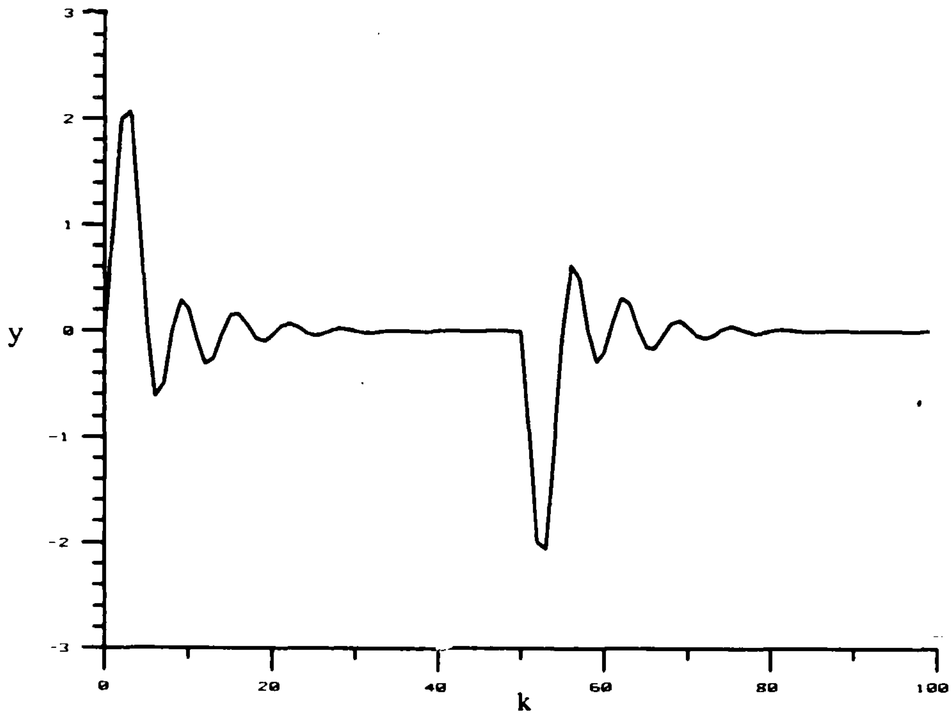

Fig. 5.3. The response of the y over one period of d for $\theta = [\,5.4 \quad 10.8 \quad 3.7\,]^T$, the value which minimizes avg[y^2].

$y(k)$ was indeed the output of a second order system with two zeros and input $d(k)$, then it would satisfy

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_1 d(k-1) + b_2 d(k-2) + b_3 d(k-3) \quad . \tag{5.12}$$

Because of the integral feedback, one of the zeros must be at $z = 1$ which implies that $b_1+b_2+b_3 = 0$. This, in turn, implies that the response to a step at $k = k_o$ satisfies

$$0 = y(k)+a_1 y(k-1)+a_2 y(k-2) \tag{5.13}$$

for all $k \geq k_o + 3$. Hence, using the equation error

$$e(k) = y(k)+a_1 y(k-1)+a_2 y(k-2) \quad . \tag{5.13}$$

we can incorporate a reference model into our cost functional. With some experimentation, we found that for the nominal plant the cost functional

$$J = \text{avg}[\, y^2(\cdot)\,] + \alpha\, \text{avg}[\, e^2(\cdot)\,] \tag{5.14}$$

with the relative weighting $\alpha = 100$ and reference model coefficients $a_1 = -1.5$ and $a_2 = 0.5725$ has a minimum at $\theta^* = [\ 3.0 \quad 3.5 \quad 1.34\ ]^T$, which provides the good response shown in Fig.5.4. The eigenvalues of the closed-loop system (5.8) all have magnitudes less than 0.80.

5.4. Tuning of All Possible Plants

The next step in the design of a slowly adapting control system for (5.1) is to verify that the controller can be tuned for all possible values of the plant parameters and to check that the minimum of J provides a good controller parameter setting. An exhaustive search over the range of plant parameter variations reveals that the closed-loop system (5.1) with controller parameter fixed at $\theta = \theta^* = [\ 3.0 \quad 3.5 \quad 1.34\ ]^T$ is exponentially stable for all possible plants. This is important for the application of a slowly adapting controller because it suggests that we may be able to

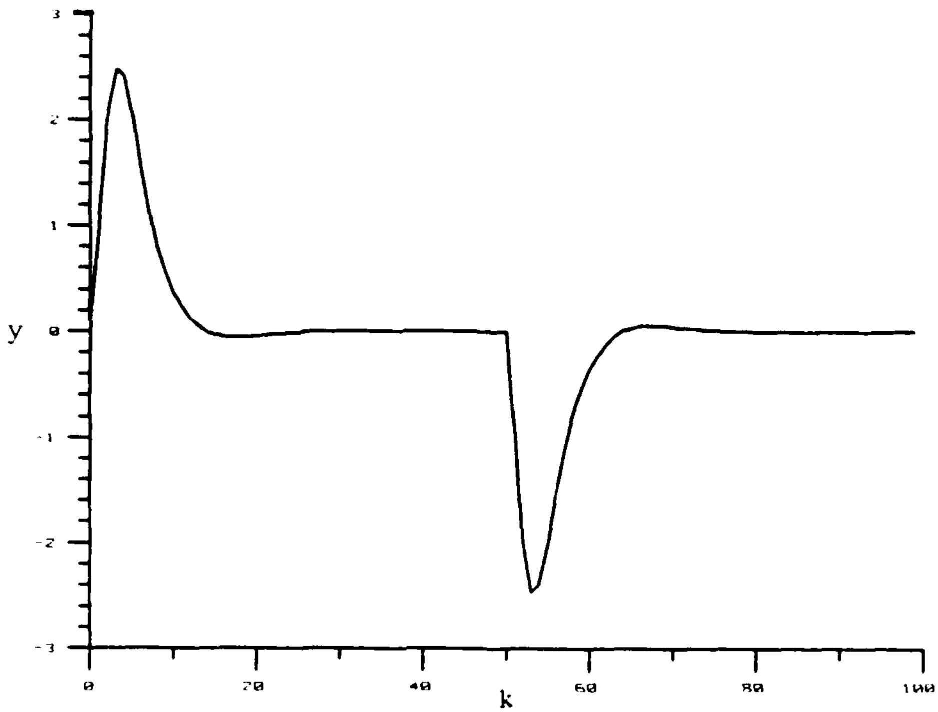

Fig. 5.4. The response of y over one period of d for $\theta = [\ 3.0 \quad 3.5 \quad 1.34\ ]^T$, the value which minimizes J given by (5.14) with $\alpha = 100$, $a_1 = -1.5$, $a_2 = 0.5725$.

initialize our controller at this value of θ for all possible plants. Although the closed-loop system has all its eigenvalues inside the unit circle for all possible plants, the response is not good for all possible plants. The extreme cases are generated by letting the plant parameter p^T take the values [0.871 0.0221 0.975], [0.871 0.0119 0.525], [0.469 0.0119 0.525], and [0.469 0.0221 0.975]. From Fig. 5.5, where the responses for the nominal value of p and each of these extreme values of p are shown, it is clear that the plant parameter variations are significant enough to require retuning of the controller. By tuning the system, we verified that the given controller is compatible and that the same cost functional which was used to tune the nominal plant can be used to tune all possible plants. The tuned responses, which are shown in Fig. 5.6, are very good. The values of θ that minimized J and tuned the control system are given in Table 5.1. For each tuned system the eigenvalues all have magnitudes less than 0.88.

5.5. Simulation Results for an Implementable Algorithm

As mentioned before, we create an implementable algorithm by using the nominal value of the plant parameter vector $p = [\,0.67 \;\; 0.017 \;\; 0.75\,]^T$ and the corresponding value of the controller parameter vector $\theta = [\,3.0 \;\; 3.5 \;\; 1.34\,]$ in the sensitivity model (5.9). In order to differentiate this approximate gradient from the true gradient, we replace v by ζ as the output of (5.9). Then, the parameter update law is given by

Table 5.1. The nominal plant parameter values and 4 sets of plant parameters which represent extreme changes from nominal along with the corresponding value of the controller parameter after tuning to minimize the cost functional J in (5.14) with $\alpha = 100$, $a_1 = -1.5$, and $a_2 = 0.5725$.

p_1	p_2	p_3	θ_1	θ_2	θ_3
0.670	0.0170	0.750	3.0	3.5	1.34
0.871	0.0221	0.975	2.3	2.2	0.40
0.871	0.0119	0.525	4.4	5.3	0.73
0.469	0.0119	0.525	4.4	5.2	3.1
0.469	0.0221	0.975	2.3	2.5	1.67

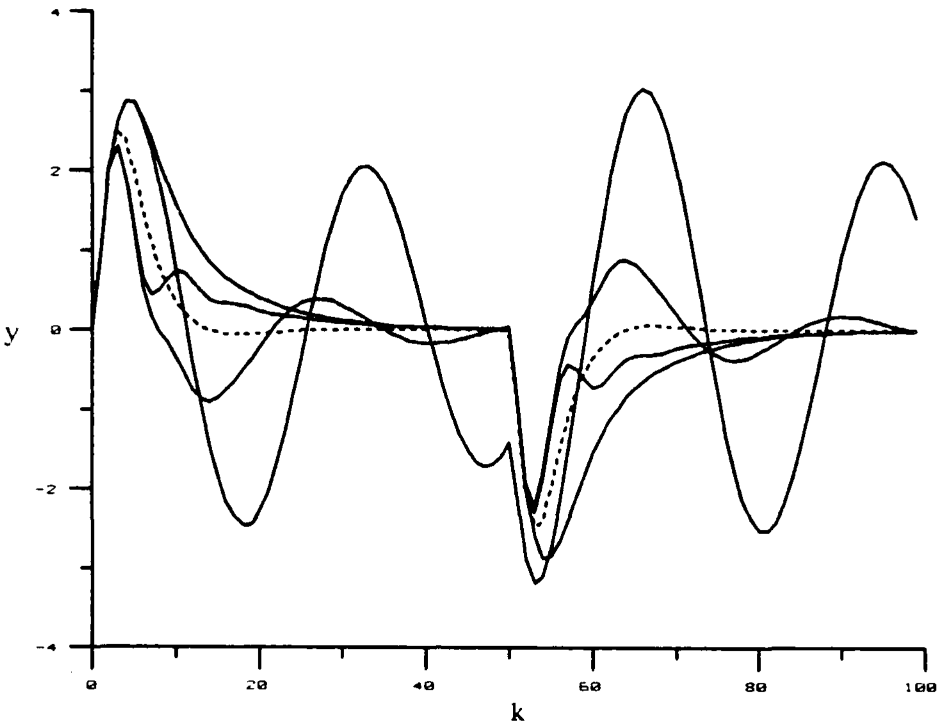

Fig. 5.5. The responses for $\theta = [\ 3.0\quad 3.5\quad 1.34\ ]^T$ and different values of the plant parameter vector p. The dashed response is for the nominal value of p.

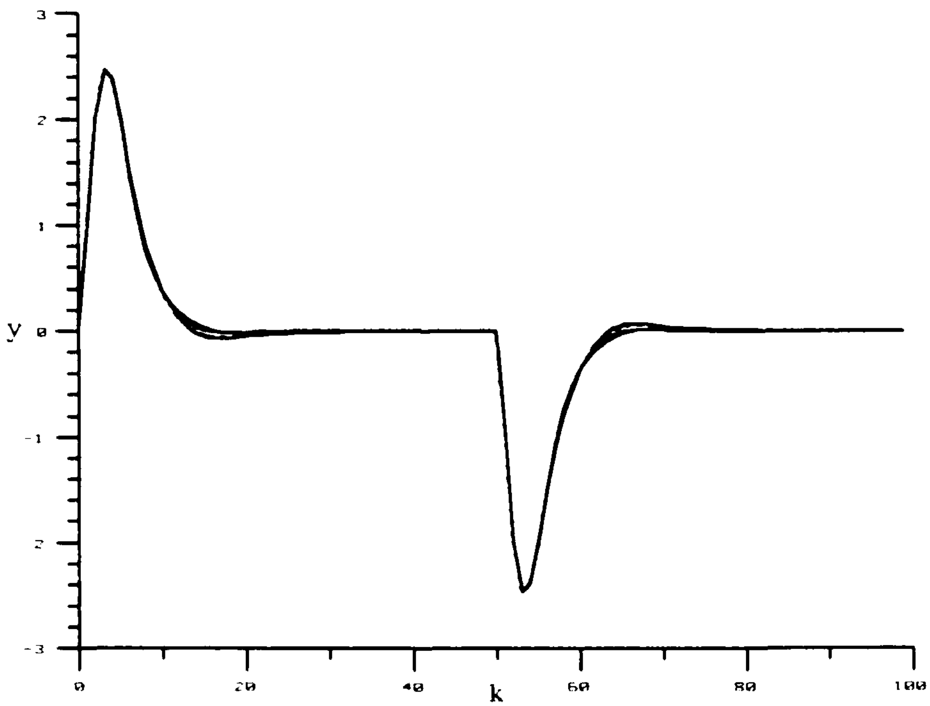

Fig. 5.6. The responses for the same values of p as in Fig. 5.5 after retuning θ. The values of p and θ are given in Table 5.1.

$$\theta(k+1) = \theta(k) - \epsilon\Gamma[\zeta(k)y(k) + 100[\zeta(k) - 1.5\zeta(k-1) + .5725\zeta(k-2)]e(k)] \quad . \tag{5.15}$$

By experimentation we found that $\epsilon = 0.01$ and $\Gamma = \mathrm{diag}(\ 2\ \ 30\ \ 1.5\ )$ provided good parameter convergence as illustrated by the trajectories of θ converging from its tuned value for the nominal plant to its tuned value for each of the extreme plants in Figs. 5.7-5.10. In Fig. 5.7 the controller parameters θ_1 and θ_3 converge quickly with monotonically decreasing average values, while the parameter θ_2 converges more slowly and its average moves initially in the wrong direction. For this value $p = [\ .871\ \ .0221\ \ .975\ ]^T$ the output is not very sensitive to the controller parameter θ_2. This can be seen from the fact that the output changes very little after k=500, but θ_2 does not converge until after k=1000. In Fig. 5.8 all three parameters converge very quickly. The averages of θ_1 and θ_3 are again monotonic, while the average of θ_2 overshoots slightly its tuned value before converging slowly to the value predicted in Table 5.1. The response with this value $p = [\ .871\ \ .0119\ \ .525\ ]^T$ and θ constant at its nominal value $\theta = [\ 3.0\ \ 3.5\ \ 1.34\ ]^T$ is the large magnitude oscillatory response in Fig. 5.5. Notice that with adaptation the response of y to the change in d at k=50 is almost the tuned response. This is the plant parameter change to which the controller is most sensitive; hence, it is the one used to tune the gain matrix Γ in the parameter update law. The diagonal elements of Γ were chosen as large as possible without causing the parameters to significantly overshoot the tuned values. The convergence of the parameters shown in Fig. 5.9 and Fig. 5.10 is about the same speed as that in Fig. 5.7, which is slower than that in Fig. 5.8. This indicates that the controller is less sensitive to these plant parameter changes.

5.6. Analysis of the Implementable Algorithm

Suppose that p is given and fixed. Let θ^* be the value of θ which tunes the controller for the given value of p. We denote by p^n and θ^n the nominal value of p and the associated tuned value of θ. By design, all of the assumptions for the existence of an exponentially attractive integral manifold are satisfied in some ball around θ^* in the parameter space. In order to analyze the behavior in the manifold, we apply averaging theory. We could analyze the slowly adapting system (5.8),(5.9),(5.15) by first showing that ζ is a good approximation of the true gradient v, and

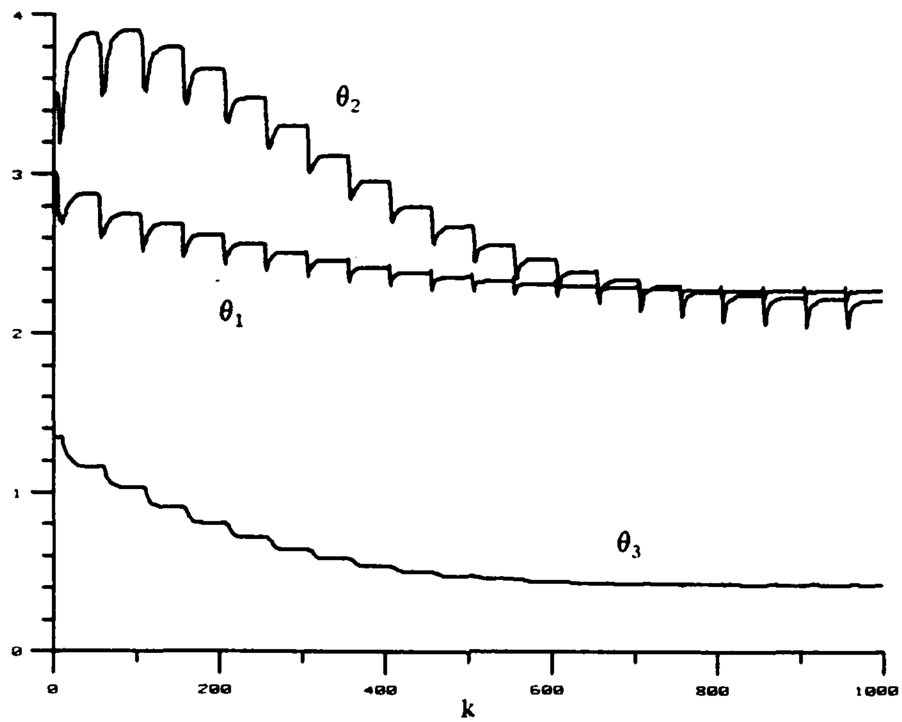

Fig. 5.7a. The controller parameters converging from nominal values to the tuned values for the plant parameter vector p = [.871 .0221 .975].

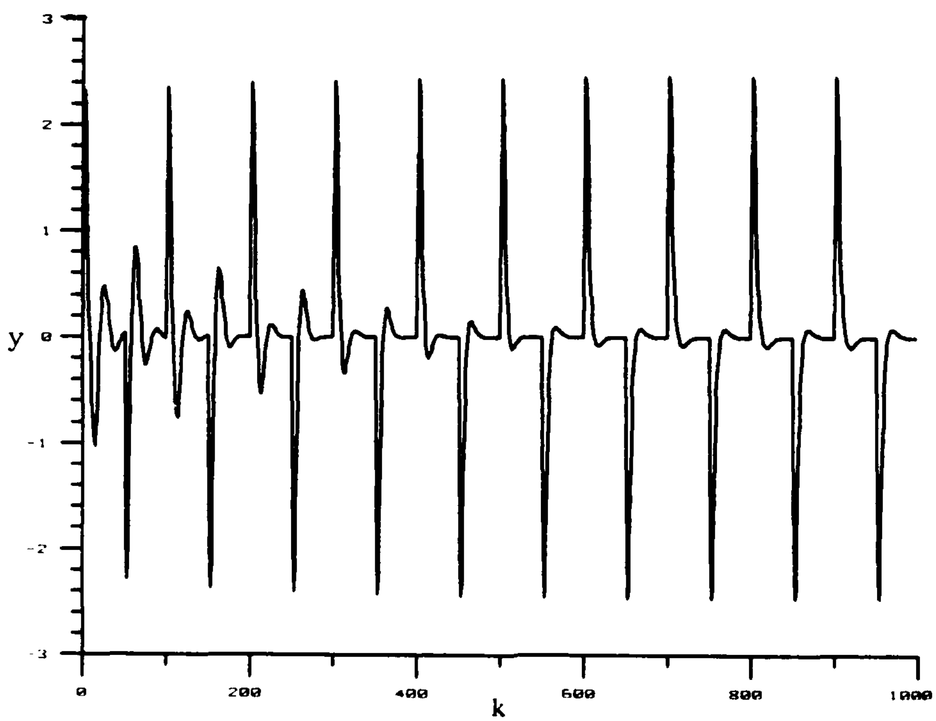

Fig. 5.7b. The output y during this tuning transient.

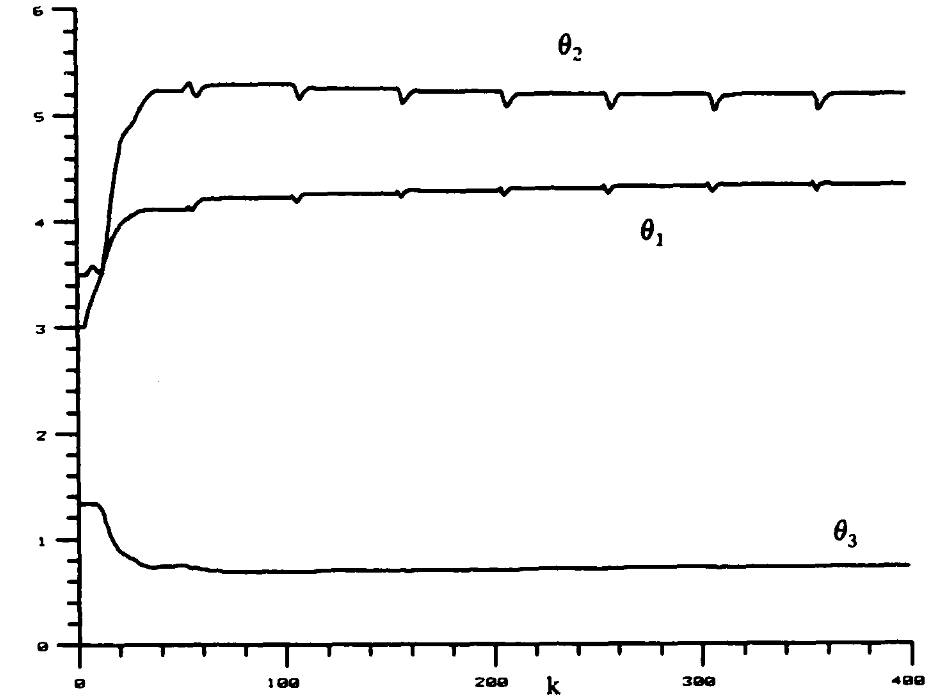

Fig. 5.8a. The controller parameters converging from nominal values to the tuned values for the plant parameter vector p = [.871 .0119 .525].

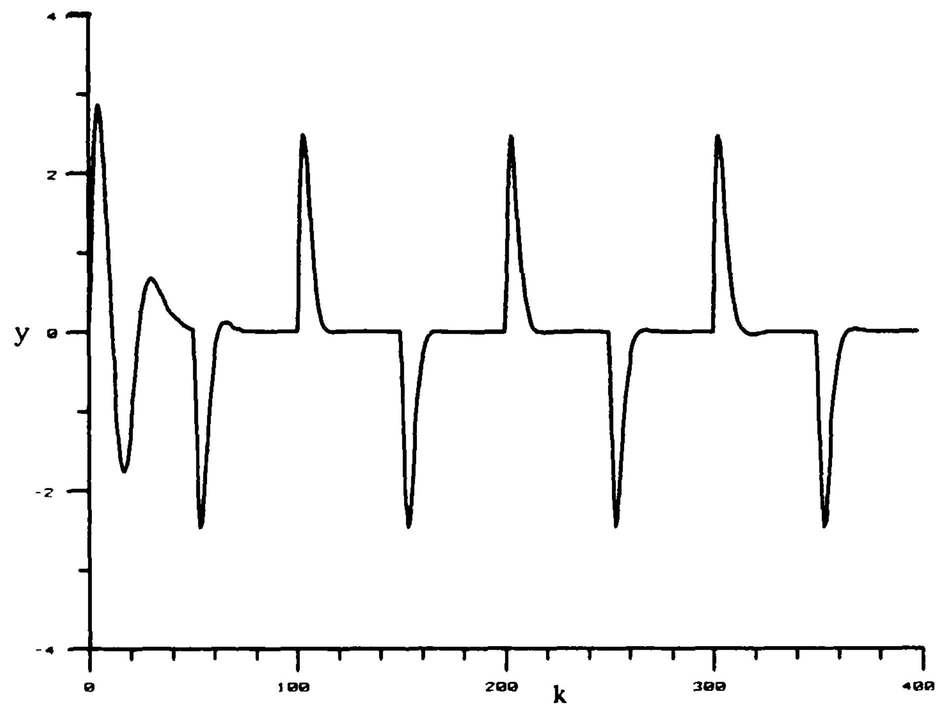

Fig. 5.8b. The output y during this tuning transient.

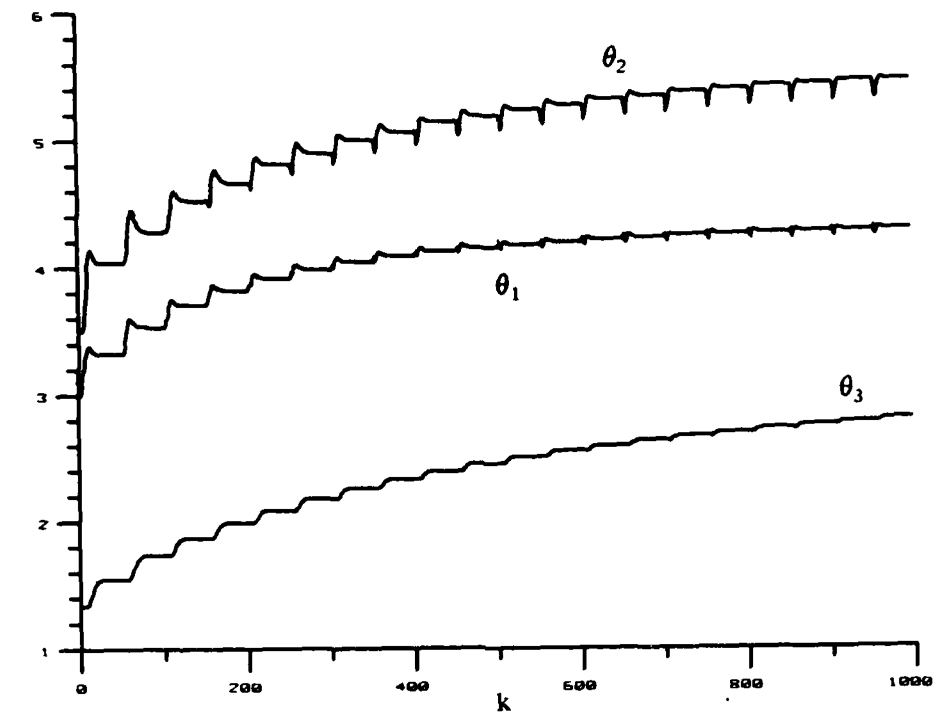

Fig. 5.9a. The controller parameters converging from nominal values to the tuned values for the plant parameter vector p = [.469 .0119 .525].

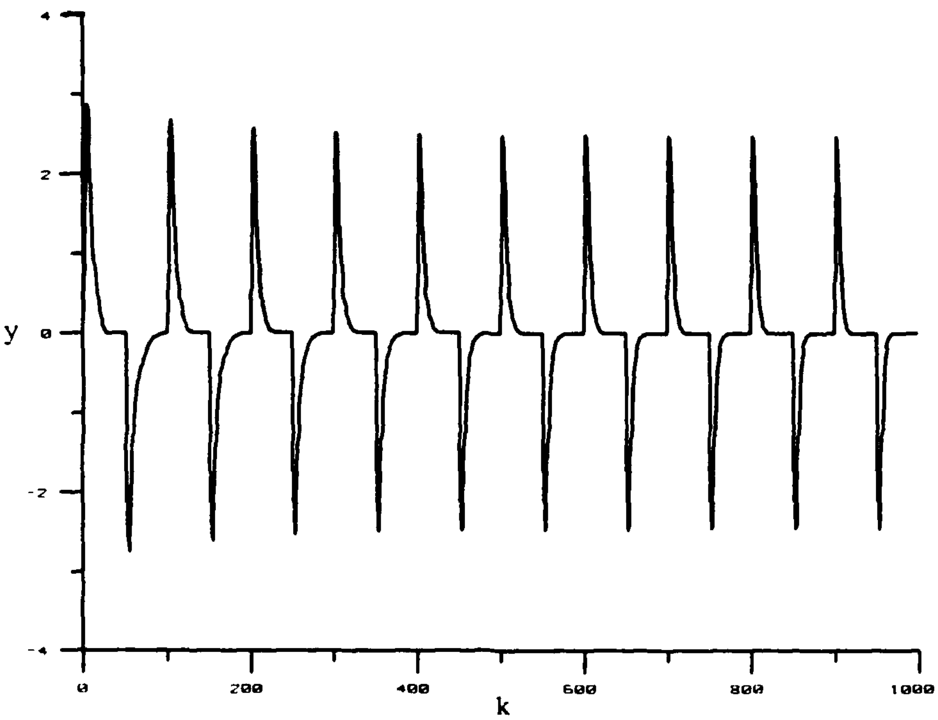

Fig. 5.9b. The output y during this tuning transient.

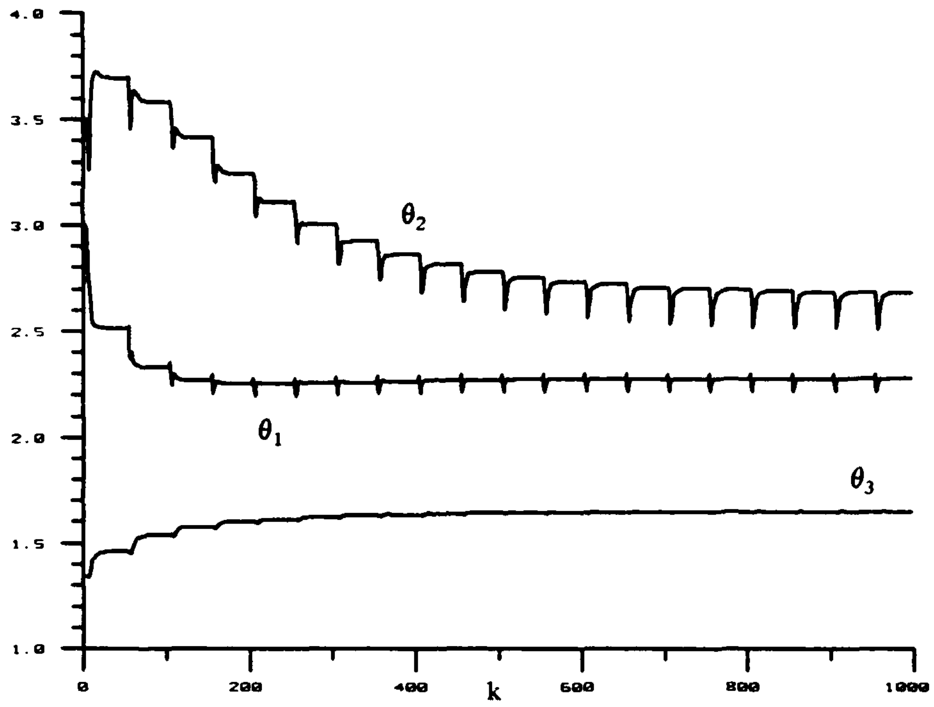

Fig. 5.10a. The controller parameters converging from nominal values to the tuned values for the plant parameter vector p = [.469 .0221 .975].

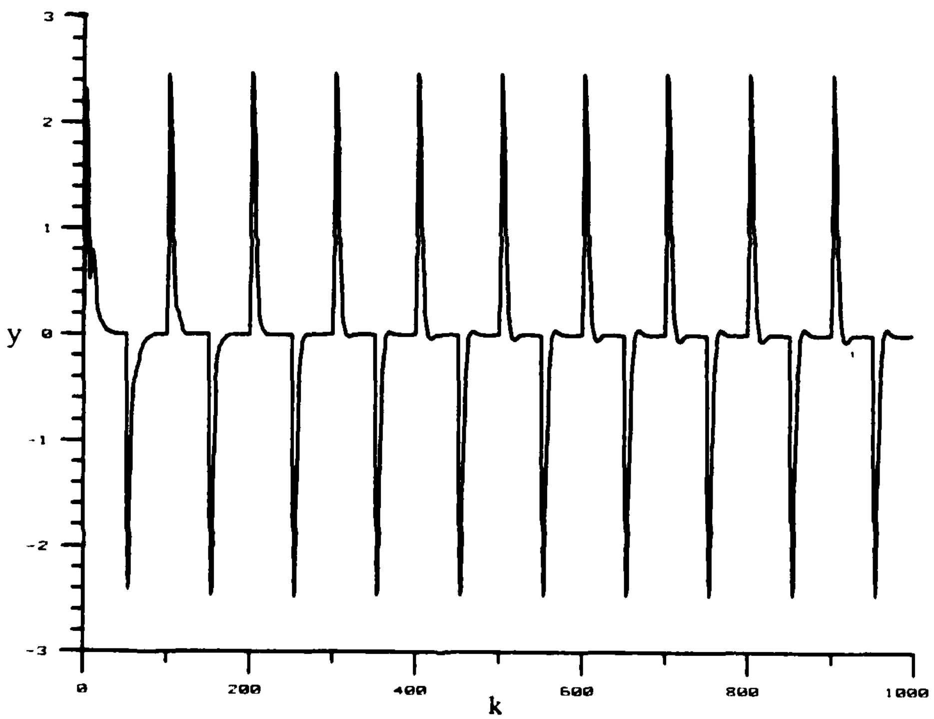

Fig. 5.10b. The output y during this tuning transient.

then, using $J(\theta)$ as a Lyapunov function to quarantee that the solutions of the ODE converge to a neighborhood of θ^*. This approach relies heavily on the knowledge of $J(\theta)$ which can be evaluated numerically but is difficult to describe analytically. We choose, instead, to follow the analysis in Chapter 4. Letting

$$\zeta_e(k) = \zeta(k) - 1.5\zeta(k-1) + .5725\zeta(k-2) \ , \tag{5.16}$$

the averaged system is described by

$$\overline{f}(\theta) = -\mathrm{avg}[\ \zeta(\cdot)y(\cdot)\ ] - \alpha\,\mathrm{avg}[\ \zeta_e(\cdot)e(\cdot)\ ] \ . \tag{5.17}$$

Defining

$$\zeta(k,p,\theta,p^n,\theta^n) = \begin{bmatrix} -10 \\ -\dfrac{.5z+.2}{z^2-.4z+.07}\,\dfrac{.5(z+1)}{z-p_1^n} \\ -\dfrac{.5z+.2}{z^2-.4z+.07}\,\dfrac{.5(z+1)}{z-p_1^n}\,\dfrac{1}{z-1} \end{bmatrix} \frac{p_2^n}{z} W_{CL}(p^n,\theta^n,z)y(k,p,\theta) \ . \tag{5.18}$$

where W_{CL} is the transfer function from d to y.

$$W_{CL}(p,\theta,z) = \frac{\dfrac{1}{z-.994}}{1+\dfrac{p_2}{z}\dfrac{1}{z-.994}\left[10\theta_1+\dfrac{.5(z+1)}{z-p_1}\left[p_3+\left[\theta_2+\dfrac{\theta_3}{z-1}\right]\dfrac{.5z+.2}{z^2-.4z+.07}\right]\right]} \ . \tag{5.19}$$

we rewrite (5.17) in the form

$$\overline{f}(\theta) = -R(p,\theta,p^n,\theta^n,\theta^*)(\theta-\theta^*) - b(p,\theta,p^n,\theta^n,\theta^*) \tag{5.20}$$

with

$$\begin{aligned} R(p,\theta,p^n,\theta^n,\theta^*) = {} & \mathrm{avg}[\ \zeta(\cdot,p,\theta,p^n,\theta^n)\zeta^T(\cdot,p,\theta,p,\theta^*)\ ] \\ & + \alpha\,\mathrm{avg}[\ \zeta_e(\cdot,p,\theta,p^n,\theta^n)\zeta_e^T(\cdot,p,\theta,p,\theta^*)\ ] \end{aligned} \tag{5.21}$$

$$\begin{aligned} b(p,\theta,p^n,\theta^n,\theta^*) = {} & \mathrm{avg}[\ \zeta(\cdot,p,\theta,p^n,\theta^n)y(\cdot,p,\theta^*)\ ] \\ & + \alpha\,\mathrm{avg}[\ \zeta_e(\cdot,p,\theta,p^n,\theta^n)e(\cdot,p,\theta^*)\ ] \ . \end{aligned} \tag{5.22}$$

The analysis then proceeds as in Chapter 4 with Theorem 4.2 providing a sufficient condition for

the exponential stability of an invariant set containing θ^*. From the closeness of the tuned responses in Fig. 5.6, we conclude that $W_{CL}(p^n,\theta^n,z)y(k,p,\theta) \cong W_{CL}(p,\theta^*,z)y(k,p,\theta)$ for all θ in a ball around θ^*. Then, the difference between $\zeta(k,p,\theta,p^n,\theta^n)$ and $\zeta(k,p,\theta,p,\theta^*)$ is due to the difference in the transfer functions $\frac{1}{z-p_1^n}$ and $\frac{1}{z-p_1}$. Since ζ_e is simply a moving average of ζ, the difference between $\zeta_e(k,p,\theta,p^n,\theta^n)$ and $\zeta_e(k,p,\theta,p,\theta^*)$ is also due to the difference in these two transfer functions. The fact that p_1 varies no more than 30% from p_1^n ensures that the matrix $R + R^T$ is positive semidefinite. From the convergence of the parameters in Figs. 5.7-5.10, we conclude that ζ is persistently exciting for the three controller parameters. This implies that $R + R^T$ is in fact positive definite. Hence R satifies the hypotheses of Theorem 4.2 and our analysis agrees with our simulations. We remark that with the computer-aided design tools available today it is more efficient to estimate via simulation and other numerical tests the size of the balls around θ^* which arise in the analysis than to estimate these balls analytically.

5.7. Concluding Remarks

In this chapter we designed a slowly adapting control system for a given plant with uncertain parameters and a given controller parametrization. We illustrated the use of an equation error approach for including reference model information in the cost functional used for tuning the system. Using our guidelines for the development of a slowly adapting system, we were able to make use of *a priori* information in the design, analysis, and testing phases of the development. We point out that many of the steps involved in developing a slowly adapting control system are already included in the design of fixed parameter control systems. Finally, we emphasize that by taking advantage of slow adaptation, we can develop adaptive control systems for controllers with given structures. Hence, existing fixed gain control systems can be upgraded to slowly adapting control systems without reparametrization.

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VITA

Bradley Dean Riedle was born in Vandalia, Illinois, on February 23, 1960. Raised in Centralia, Illinois, he graduated 4th in a class of 350 from Centralia Township High School in 1978. Since then, he has attended the University of Illinois at Urbana-Champaign. He received the B. S. degree with highest honors in Electrical Engineering in 1982, and the M. S. degree in Electrical Engineering in 1984.

Dr. Riedle has been the recipient of numerous awards and fellowships during his academic career. As an undergraduate, he was a Kodak Scholar and was awarded All University Honors and the E. C. Jordan Award for academic excellence in Electrical Engineering. As a graduate student, he has been an Exxon Fellow, an Allied Fellow, and has received a departmental fellowship from the Electrical and Computer Engineering Department.

Dr. Riedle is the coauthor of one book, six papers in reviewed journals, and nine papers which were (or shall be) presented at conferences or workshops with published proceedings. The book and four of the journal papers are referenced in this thesis. The more important of the two remaining journal papers is an article written with Prof. Petar V. Kokotovic and entitled "Stability Analysis of an Adaptive System with Unmodeled Dynamics." It appeared in the *International Journal of Control* in 1985.

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